POLYHEDRAL BOUNDS ON THE JOINT SPECTRUM AND TEMPEREDNESS OF LOCALLY SYMMETRIC SPACES

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Abstract. Given a real semisimple connected Lie group G and a discrete torsion-free subgroup $\Gamma < G$ we prove a precise connection between growth rates of the group Γ, polyhedral bounds on the joint spectrum of the ring of invariant differential operators, and the decay of matrix coefficients. In particular, this allows us to completely characterize temperedness of $L^2(\Gamma\backslash G)$ in this general setting.

1. Introduction

Consider a locally symmetric space $\Gamma\backslash G/K$, where G is a real connected semisimple non-compact Lie group with finite center, K is a maximal compact subgroup, and $\Gamma < G$ is a discrete torsion-free subgroup. There is a general scheme to connect the spectral properties of $\Gamma\backslash G/K$ with growth rates of the discrete group Γ. One of the first instances of this connection is the characterization of the bottom $\inf \sigma(\Delta)$ of the Laplace spectrum for hyperbolic surfaces:

$$\inf \sigma(\Delta) = \begin{cases} 1/4 & : \delta_\Gamma < 1/2 \\ 1/4 - (\delta_\Gamma - 1/2)^2 & : \delta_\Gamma \geq 1/2, \end{cases}$$

where δ_Γ is the critical exponent of the discrete subgroup $\Gamma \leq SL_2(\mathbb{R})$

$$\delta_\Gamma := \inf \left\{ s \in \mathbb{R} \colon \sum_{\gamma \in \Gamma} e^{-sd(\gamma x_0, x_0)} < \infty \right\}, \quad x_0 \in \mathbb{H}.$$

This theorem is due to Elstrodt [Els73a, Els73b, Els74] and Patterson [Pat76] and has been extended to real hyperbolic manifolds of arbitrary dimension by Sullivan [Sul87] and then to general locally symmetric spaces of rank one by Corlette [Cor90].

We are interested in analogous statements for higher rank locally symmetric spaces. An important feature is the following: G admits a Cartan decomposition $G = K \exp(\overline{\mathfrak{a}_+})K$. Hence, for every $g \in G$ there is $\mu_+(g) \in \overline{\mathfrak{a}_+}$ such that $g \in K \exp(\mu_+(g))K$. $\mu(g)$ can be thought of a higher dimensional distance $d(gK, eK)$.

In this higher rank setting the bottom of the Laplace spectrum was estimated using the same definition of δ_Γ which is defined through $d(\gamma x_0, x_0) = \|\mu_+(x_0^{-1}\gamma x_0)\|$ [Web08, Leu04]. Later, Anker and Zhang [AZ22] (see also [CP04]) proved the exact formula

$$\inf \sigma(\Delta) = \begin{cases} \|\rho\|^2 & : \tilde{\delta}_\Gamma < \|\rho\| \\ \|\rho\|^2 - (\tilde{\delta}_\Gamma - \|\rho\|)^2 & : \tilde{\delta}_\Gamma \geq \|\rho\|, \end{cases}$$

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where ρ is the usual half sum of restricted roots and $\tilde{\delta}_\Gamma$ is the modified critical exponent which is defined through $\|\mu_+(\gamma)\|$ and $\langle \rho, \mu_+(\gamma)\rangle$ and therefore also takes the direction and not only the size of $\mu_+(\gamma)$ into account.

This concept can be further extended through the definition of the growth indicator function $\psi_\Gamma\colon \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ introduced by Quint [Qui02]:

$$\psi_\Gamma(H) \coloneqq \|H\| \inf_{H\in\mathcal{C}} \inf\{s \in \mathbb{R} \mid \sum_{\gamma\in\Gamma, \mu_+(\gamma)\in\mathcal{C}} e^{-s\|\mu_+(\gamma)\|} < \infty\},$$

where the infimum runs over all open cones $\mathcal{C} \subseteq \mathfrak{a}$ with $H \in \mathcal{C}$. It has been shown in [WZ23] that

$$\inf \sigma(\Delta) = \|\rho\|^2 - \max\left\{0, \sup_{H\in\overline{\mathfrak{a}_+}} \frac{\psi_\Gamma(H) - \langle \rho, H\rangle}{\|H\|}\right\}^2.$$

In the rank one case an immediate consequence of the above described relations is that the representation $L^2(\Gamma\backslash G)$ is tempered if and only if $\delta_\Gamma \le 1/2$. This follows, because in rank one all non-tempered representations occurring in $L^2(\Gamma\backslash G)$ lead to Laplace eigenvalues strictly smaller than $\|\rho\|$. The latter argument breaks down completely in higher rank, as there are known examples of non-tempered representations that lead to arbitrary high Laplace eigenvalues. Thus the question of temperedness of $L^2(\Gamma\backslash G)$ remained completely open until the recent breakthrough of Edwards and Oh who proved the following theorem, based on previously obtained mixing results for Anosov subgroups [ELO23]:

Theorem 1 ([EO23, Theorem 1.6])**.**
(i) If $L^2(\Gamma\backslash G)$ is tempered then $\psi_\Gamma \le \rho$.
(ii) Assume that Γ is a Zariski dense image of an Anosov representation with respect to the minimal parabolic subgroup. Then $\psi_\Gamma \le \rho$ implies that $L^2(\Gamma\backslash G)$ is tempered.

A consequence of the main result in the present paper is that this result holds for general discrete subgroups $\Gamma < G$ (see Corollary 3). We can deduce this result from a general polyhedral bound on the joint spectrum $\widetilde{\sigma} \subseteq \mathfrak{a}_\mathbb{C}^*/W$ of the algebra of invariant differential operators on G/K (see Section 2.3 for a precise definition).

The temperedness of $L^2(\Gamma\backslash G)$ is equivalent to $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$ and in general $\Re\widetilde{\sigma} \subseteq \operatorname{conv}(W\rho)$, where $\operatorname{conv}(W\rho)$ is the polyhedron described by the convex hull of the Weyl orbit of ρ. Our main theorem states (in part) that bounding the growth indicator function ψ_Γ by dilates of the linear functional ρ is equivalent to bounding $\Re\widetilde{\sigma}$ in a dilation of the polyhedron $\operatorname{conv}(W\rho)$:

Theorem 2. *Let G be a real semisimple connected non-compact Lie group with finite center and $\Gamma < G$ a discrete and torsion-free subgroup. Then for all $p \in 2\mathbb{N}$ the following statements are equivalent:*
(i) $\Re\widetilde{\sigma} \subseteq (1 - 2p^{-1}) \operatorname{conv}(W\rho)$.
(ii) For all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for all $f_1, f_2 \in L^2(\Gamma\backslash G)^K$:

$$|\langle(\exp v) f_1, f_2\rangle| \le d_\varepsilon e^{\varepsilon\|v\|} e^{-2p^{-1}\rho(v)} \|f_1\|\|f_2\|.$$

(iii) $\psi_\Gamma \le (2 - 2p^{-1})\rho$.
(iv) $L^2(\Gamma\backslash G)$ is almost L^p (see Section 2.2 for a definition).

With
(v) $\inf \sigma(\Delta) \ge 2p^{-1}(2 - 2p^{-1})\|\rho\|^2$.

we have the following implications between the above statements for $p \in [2, \infty)$:

$$(i) \iff (ii) \implies (iii) \implies (iv), (v).$$

A direct consequence of taking $p = 2$ and [WZ23, Cor. 1.2] is:

Corollary 3. *Let G be a real semisimple connected non-compact Lie group with finite center and $\Gamma < G$ a discrete and torsion-free subgroup, then the following statements are equivalent:*

(i) $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$.

(ii) For all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for all $f_1, f_2 \in L^2(\Gamma\backslash G)^K$:

$$|\langle(\exp v)f_1, f_2\rangle| \leq d_\varepsilon e^{\varepsilon\|v\|} e^{-\rho(v)} \|f_1\| \|f_2\|.$$

(iii) $\psi_\Gamma \leq \rho$.

(iv) $L^2(\Gamma\backslash G)$ *is almost* L^2.

(v) $\inf \sigma(\Delta) = \|\rho\|^2$.

(vi) $L^2(\Gamma\backslash G)$ *is tempered.*

Strategy of proof. The key step in our proof is that we can derive a precise relation between the decay of matrix coefficients for functions $f_1, f_2 \in C_c(\Gamma\backslash G)$ and the growth indicator function ψ_Γ (Theorem 10). We can then link these decay estimates to the joint spectrum by the abstract Plancherel formula and the asymptotic analysis of spherical functions.

Related results. In a previous work, the latter two named authors [WW23] had obtained bounds on the joint spectrum by counting of Γ points in the case where G is a product of rank one groups and $\Gamma < G$ a general discrete, torsion free subgroup. In particular, they obtained Theorem 1 in this case. The methods in [WW23] however were based on analyzing the resolvent kernels on the individual rank one factors.

Temperedness in the complementary setting of homogeneous spaces G/H for a closed subgroup H with finitely many connected components has been studied by Benoist and Kobayashi in a series of papers [BK15, BK22, BK21, BK23]. They prove that the regular representation of G on $L^2(G/H)$ is tempered if and only if a growth condition on H is satisfied that is similar to (iii). They also prove a version alike Theorem 2 where they characterize when $L^2(G/H)$ is almost L^p for $p \in 2\mathbb{N}$.

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2. Preliminaries

2.1. **Notation.** In this article G is a real semisimple connected non-compact Lie group with finite center and K is a maximal compact subgroup of G. We fix an Iwasawa decomposition $G = KAN$ and define M as the centralizer of A in K. Furthermore, let $\overline{N}$ be the nilpotent subgroup such that $KA\overline{N}$ is the opposite Iwasawa decomposition. We denote by $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{m}, \overline{\mathfrak{n}}$ the corresponding Lie algebras. For $g \in G$ let $H(g) \in \mathfrak{a}$ be the logarithm of the A-component in the Iwasawa decomposition. Let $\Sigma \subseteq \mathfrak{a}^*$ be the root system of restricted roots, Σ^+ the positive system corresponding to the Iwasawa decomposition, and W the corresponding Weyl group acting on $\mathfrak{a}^*$. As usual, for $\alpha \in \Sigma$, we denote by m_α the dimension of the root space, and by ρ the half sum of restricted roots counted with multiplicity. Let $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \,\forall \alpha \in \Sigma\}$ the positive Weyl chamber, $\overline{\mathfrak{a}_+}$ its closure, and $\mathfrak{a}^*_+$ the corresponding cone in $\mathfrak{a}^*$ via the identification $\mathfrak{a} \leftrightarrow \mathfrak{a}^*$ through the Killing form $\langle \cdot, \cdot \rangle$. We have the Cartan decomposition $G = K \exp(\overline{a_+})K$ and for $g \in G$ there is a unique $\mu_+(g) \in \overline{a_+}$ such that $g \in K \exp(\mu_+(g))K$. For the Cartan decomposition the following integral formula holds (see [Hel84, Thm. I.5.8]):

$$\int_G f(g)\, dg = \int_K \int_{\mathfrak{a}_+} \int_K f(k \exp(H) k') \delta(H)\, dk\, dH\, dk' \tag{2.1}$$

where $\delta(H) = \prod_{\alpha \in \Sigma^+} (\sinh(\alpha(H))^{m_\alpha}$. Note that $\delta(H) \leq e^{2\rho(H)}$. We fix a discrete subgroup $\Gamma \leq G$.

2.2. **Temperedness and almost L^p.** Recall the following definitions. Denote by Ξ the Harish-Chandra function $\Xi(g) = \int_K e^{-\rho(H(gk))}\, dk$ where $H\colon G \to \mathfrak{a}$ is defined by $g \in Ke^{H(g)}N$. It is well-known that Ξ is a smooth bi-K-invariant function of G with values in $(0, 1]$. Furthermore, there is a constant C such that

$$e^{-\rho(H)} \leq \Xi(e^H) \leq C(1 + |H|)^d e^{-\rho(H)}$$

for $H \in \mathfrak{a}_+$. Here d is the number of positive reduced roots. Note that by (2.1) this implies that $\Xi \in L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$.

Definition 2.1 ([Oh02, Def. 2.3 and 2.4])**.**
(i) A representation π of G is called *tempered* if for any K-finite unit vectors v and w,

$$|\langle \pi(g)v, w\rangle| \leq (\dim\langle Kv\rangle \dim\langle Kw\rangle)^{1/2}\, \Xi(g)$$

for any $g \in G$, where $\langle Kv\rangle$ denotes the subspace spanned by Kv.
(ii) A representation π of G is called *strongly* $L^{p+\varepsilon}$ or *almost* L^p if there is a dense subset V of the Hilbert space attached to π such that for any $v, w \in V$, the matrix coefficient $g \mapsto \langle \pi(g)v, w\rangle$ lies in $L^q(G)$ for all $q > p$.

Note that if π is strongly $L^{p+\varepsilon}$, then π is also strongly $L^{q+\varepsilon}$ for any $q \geq p$ since any matrix coefficients are bounded. We have the following theorem relating the two concepts.

Theorem 4 ([Oh02, Thm. 2.4])**.** *A representation π is tempered if and only if it is strongly $L^{2+\varepsilon}$.*

Since we're not only interested in temperedness, being strongly $L^{p+\varepsilon}$ gives us a measure for the extent of the non-tempered part. However, the connection to uniform pointwise bounds seems to be established only for $p \in 2\mathbb{N}$:

Theorem 5 ([Oh02, Thm. 2.5])**.** *If π is a unitary representation without a non-zero invariant vector that is strongly $L^{2k+\varepsilon}$, $k \in \mathbb{N}$, then for any K-finite unit vectors v and w,*

$$|\langle \pi(g)v, w\rangle| \leq (\dim\langle Kv\rangle \dim\langle Kw\rangle)^{1/2}\, \Xi^{1/k}(g).$$

Clearly, since Ξ is $L^{2+\varepsilon}$ the opposite implication holds as well.

2.3. **Spherical dual and joint spectrum.** Assume that the unitary representation R on $L^2(\Gamma\backslash G)$ by right multiplication is decomposed into a direct integral of irreducible representations

$$\pi \simeq \int_X^{\oplus} \pi_x \, d\mu(x)$$

where (X,μ) is a measure space and π_x are irreducible unitary representations. We should think of X as the Cartesian product of the unitary dual $\widehat{G}$ and a multiplicity space.

The joint spectrum can be defined as follows.

Definition 2.2. [[WW23, Prop. 3.6]]

$$\widetilde{\sigma} := \operatorname{supp}(\varphi_*\mu) \cap \widehat{G}_{sph} \subseteq \widehat{G}_{sph}, \tag{2.2}$$

where $\varphi\colon X \to \widehat{G}$ is the map $x \mapsto \pi_x$ and $\widehat{G}_{sph}$ is the spherical dual of G, i.e. the set of equivalence classes of irreducible unitary representations containing a non-zero K-invariant vector.

In the following we describe how $\widehat{G}_{sph}$ can be parametrized by subset of $\mathfrak{a}_{\mathbb{C}}^*/W$ (see [Hel84, Thm. IV.3.7]). For $\pi \in \widehat{G}_{sph}$ let v_K be a normalized K-invariant vector. Then the function $\phi\colon G \to \mathbb{C}, \phi(g) = \langle \pi(g)v_K, v_K\rangle$ is bi-K-invariant and positive definite, i.e. the matrix $(\phi(x_i^{-1}x_j))_{ij}$ is positive semidefinite for any choice of finitely many $x_i \in G$. Furthermore, ϕ is an eigenvector for each element in the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K. Therefore, ϕ is an elementary spherical function $\phi_\lambda(g) = \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\, dk$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Note that $\phi_\lambda = \phi_\mu$ if and only if $W\lambda = W\mu$. It can be shown that the mapping $\pi \mapsto W\lambda$ is a bijection of $\widehat{G}_{sph}$ onto the set $\{\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W \mid \phi_\lambda \text{ is positive definite}\}$. We identify the two sets and write π_λ for the representation corresponding to $\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W$ with ϕ_λ positive definite. In particular, for $\lambda \in \widehat{G}_{sph}$ we have $\langle \pi_\lambda(g)v, w\rangle = \phi_\lambda(g)\langle v, w\rangle$ if v, w are K-invariant.

Every positive definite function on G is bounded by its value at 1 and therefore $\widehat{G}_{sph} \subseteq \operatorname{conv}(W\rho) + i\mathfrak{a}^*$ by [Hel84, Thm. IV.8.1]. Here $\operatorname{conv}(W\rho)$ is the convex hull of the Weyl orbit $W\rho$ of ρ which can be characterized by

$$\operatorname{conv}(W\rho) = \{\lambda \in \mathfrak{a}^* \mid \lambda(wH) \le \rho(H)\ \forall H \in \mathfrak{a}_+, w \in W\}.$$

$L^2(\Gamma\backslash G)$ is tempered if and only if $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$ (see Lemma 9 below) and the intermediate cases will be covered by the least $p \in [2,\infty]$ such that $\Re\widetilde{\sigma} \subseteq (1-2p^{-1})\operatorname{conv}(W\rho)$.

3. Decay of coefficients in the L^2 sense

In this section we prove series of lemmas that connect the decay of coefficient in the L^2 sense with the growth indicator function ψ_Γ as well as the joint spectrum. We start with a slight modification of [LO23, Prop. 7.3].

Lemma 6. *Suppose that there exists a homogeneous function $\theta\colon \mathfrak{a}_+ \to \mathbb{R}$ such that for any $\varepsilon > 0$ there is $d_\varepsilon > 0$ such that for any K-invariant functions $f, g \in L^2(\Gamma\backslash G)^K$, any $v \in \mathfrak{a}_+$,*

$$|\langle R(\exp v)f, g\rangle| \le d_\varepsilon e^{-\theta(v)} e^{\varepsilon\|v\|}\|f\|_2\|g\|_2.$$

Then this implies

$$\psi_\Gamma \le 2\rho - \theta.$$

Note that it is enough to have the assumption for $f, g \in C_c(\Gamma\backslash G)^K$ since C_c is dense in L^2.

Proof. The proof is the same as [LO23, Prop. 7.3] except that one obtains

$$\#(\Gamma \cap B_T) \leq Ce^{(T+\varepsilon)((2\rho-\theta)u+\varepsilon\|u\|)+2(T+\varepsilon)\eta}.$$

Therefore,

$$\limsup_{T\to\infty} \frac{1}{T} \log \#(\Gamma \cap B_T) \leq (2\rho - \theta)(u) + \varepsilon\|u\| + 2\eta.$$

This implies $\psi_\Gamma(u) \leq (2\rho - \theta)(u) + \varepsilon\|u\|$ and the lemma by letting $\varepsilon \to 0$. □

We now prove that the assumption of Lemma 6 is satisfied by a homogeneous functional θ defined via the joint spectrum.

Lemma 7. *For all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for all $f, g \in L^2(\Gamma\backslash G)^K$ we have*

$$\langle R(\exp v)f, g\rangle \leq d_\varepsilon e^{\sup_{\lambda\in\widetilde{\sigma}}(\Re\lambda-\rho)(v)} e^{\varepsilon\|v\|}\|f\|_2\|g\|_2.$$

Proof. We decompose $f, g \in L^2(\Gamma\backslash G)^K$ into $\int_X^\oplus f_x \, d\mu(x)$ and $\int_X^\oplus g_x \, d\mu(x)$, respectively, according to the decomposition $L^2(\Gamma\backslash G) \simeq \int_X^\oplus \pi_x \, d\mu(x)$. Since f and g are K-invariant f_x and g_x are contained in π_x^K for almost every $x \in X$ and hence they vanish for almost every $x \in X$ with $\pi_x \notin \widehat{G}_{sph}$. We calculate

$$\langle R(\exp v)f, g\rangle = \int_X \langle \pi_x(\exp v)f_x, g_x\rangle \, d\mu(x) = \int_{\varphi^{-1}(\widehat{G}_{sph})} \langle \pi_x(\exp v)f_x, g_x\rangle \, d\mu(x).$$

We recall that if $\lambda \in \mathfrak{a}_\mathbb{C}^*/W$ corresponds to $\pi_\lambda \in \widehat{G}_{sph}$ we have

$$\langle \pi_\lambda(g)v_K, v_K\rangle = \phi_\lambda(g)\langle v_K, v_K\rangle$$

for $v_K \in \pi_\lambda^K$. Therefore,

$$\langle R(\exp v)f, g\rangle = \int_{\varphi^{-1}(\widehat{G}_{sph})} \phi_{\lambda_x}(\exp v)\langle f_x, g_x\rangle \, d\mu(x).$$

Hence we can estimate

$$\begin{aligned}
|\langle R(\exp v)f, g\rangle| &\leq \int_{\varphi^{-1}(\widehat{G}_{sph})} |\phi_{\lambda_x}(\exp v)|\|f_x\|\|g_x\| \, d\mu(x) \\
&\leq \operatorname{esssup}_{\varphi_*\mu|_{\widehat{G}_{sph}}} |\phi_{\lambda_x}(\exp v)|\|f\|_2\|g\|_2 \\
&\leq \sup_{\lambda\in\widetilde{\sigma}} |\phi_\lambda(\exp v)|\|f\|_2\|g\|_2.
\end{aligned}$$

For the elementary spherical function we have the well-known estimates

$$|\phi_\lambda(\exp v)| \leq e^{\Re\lambda(v)}\Xi(\exp v) \leq d_\varepsilon e^{\Re\lambda(v)}e^{-\rho(v)}e^{\varepsilon\|v\|}$$

for $\Re\lambda \in \mathfrak{a}_+$ and any $\varepsilon > 0$. This completes the proof. □

As a direct consequence of Lemma 6 and 7 we get the following proposition.

Proposition 8.

$$\psi_\Gamma(v) \leq \sup_{\lambda\in\widetilde{\sigma}} \Re\lambda(v) + \rho(v).$$

Note that this bound on the counting function is even a little bit more precise compared to the bounds stated in the main theorem, because the right hand side is not simply a dilation of ρ but might be a more precise functional.

We can also prove an analogue of Lemma 6 which implies an obstruction on the joint spectrum instead of an obstruction on ψ_Γ.

Lemma 9. *Suppose that there exists a homogeneous function $\theta\colon \mathfrak{a}_+ \to \mathbb{R}$ such that for all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for any K-invariant functions $f, g \in L^2(\Gamma\backslash G)$ and any $v \in \mathfrak{a}_+$*

$$|\langle R(\exp v)f, g\rangle| \le d_\varepsilon e^{-\theta(v)} e^{\varepsilon\|v\|}\|f\|_2\|g\|_2.$$

Then this implies that

$$\Re\lambda \le \rho - \theta$$

for all $\lambda \in \widetilde{\sigma}$.

In particular, $L^2(\Gamma\backslash G)$ is tempered if and only if $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$.

Proof. Let $\tilde{\varepsilon} > 0$, $X_{sph} = \varphi^{-1}(\widehat{G}_{sph})$, $\lambda_0 \in \widetilde{\sigma}$, and $A_{\tilde{\varepsilon}} := \{x \in X_{sph} \mid |\lambda_x - \lambda_0| < \varepsilon\}$. Then $\mu(A_{\tilde{\varepsilon}}) > 0$ by (2.2). Put $f_{\tilde{\varepsilon}} = \mu(A_{\tilde{\varepsilon}})^{-1/2}\int_X^\oplus \mathbb{1}_{A_{\tilde{\varepsilon}}}(x) w_x^K \, d\mu(x)$ where $w_x^K \in \pi_x^K$ is normalized. By definition $f_{\tilde{\varepsilon}} \in L^2(\Gamma\backslash G)^K$ is normalized and $\langle R(\exp v)f_{\tilde{\varepsilon}}, f_{\tilde{\varepsilon}}\rangle = \mu(A_\varepsilon)^{-1}\int_{A_\varepsilon} \phi_{\lambda_x}(\exp v)\, d\mu(x)$. We infer that $\phi_{\lambda_0}(\exp v) = \lim_{\tilde{\varepsilon}\to 0}\langle R(\exp v)f_{\tilde{\varepsilon}}, f_{\tilde{\varepsilon}}\rangle$ and therefore by the assumed bound on the matrix coeffcients we get $|\phi_{\lambda_0}(\exp v)| \le d_\varepsilon e^{-\theta(v)}e^{\varepsilon\|v\|}$ for any $\varepsilon > 0$. Without loss of generality assume $\Re\lambda_0 \in \overline{\mathfrak{a}_+^*}$. From [vdBS87, Thm. 3.5 and proof of Thm. 10.1] follows that there is a polynomial $p(t)$ such that

$$\phi_{\lambda_0}(\exp tv)p(t)^{-1}e^{-t(\lambda_0-\rho)(v)} \to 1 \quad \text{as} \quad t \to \infty.$$

Hence,

$$1 \le \limsup_{t\to\infty} d_\varepsilon |p(t)|^{-1} e^{t(-\theta(v)+\varepsilon\|v\| - \Re\lambda_0(v)+\rho(v))}$$

for any $\varepsilon > 0$. We conclude

$$-\theta(v) + \varepsilon\|v\| - \Re\lambda_0(v) + \rho(v) > 0$$

and

$$\Re\lambda_0 \le \rho - \theta.$$

This completes the proof. ☐

4. Decay of coefficients in terms of the growth indicator function

The goal of this section is to prove the following theorems.

Theorem 10. *Let $f_1, f_2 \in C_c(\Gamma\backslash G)$, $H_0 \in \overline{\mathfrak{a}_+}$ normalized, and $s > \psi_\Gamma(H_0)$, $s \ge 0$. Then there exists $\delta > 0$ and $C > 0$ such that*

$$|\langle R(\exp tH)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}| \le Ce^{t(s-2\rho(H))}$$

for all $t \ge 0$ and $H \in B_\delta(H_0)$ normalized.

Remark. If H_0 is not in the limit cone and therefore $\psi_\Gamma(H_0) = -\infty$, then

$$\langle R(\exp(tH))f_1, f_2\rangle = 0$$

for t large enough depending on H_0.

Theorem 11. *If $\psi_\Gamma \le (2 - 2p^{-1})\rho$ then $L^2(\Gamma\backslash G)$ is strongly $L^{p+\varepsilon}$.*

Proof of Theorem 11. Let us fix an arbitrary $\varepsilon > 0$. Based on Theorem 10 we will show that the matrix coefficients for functions in $C_c(\Gamma\backslash G)$ are $p + \varepsilon$ integrable. Let $f_1, f_2 \in C_c(\Gamma\backslash G)$. Since

$$\left|\int_{\Gamma\backslash G} f_1(\Gamma gh) f_2(\Gamma g)\, d\Gamma g\right| \le \int_{\Gamma\backslash G} \max_{k\in K}|f_1(ghk)| \max_{k\in K}|f_2(gk)|\, dg \tag{4.1}$$

we can assume that f_i is non-negative and right-K-invariant.

Since $\psi_\Gamma \leq (2-2p^{-1})\rho$ for any $H_0 \in \overline{\mathfrak{a}_+}$, we can find an $s_{H_0} \geq 0$ such that $\psi_\Gamma(H_0) < s_{H_0} < (2-2(p+\varepsilon)^{-1})\rho(H_0)$. Then by Theorem 10 for any $H_0 \in \overline{\mathfrak{a}_+}$ normalized, there is $\delta > 0$ and $C > 0$ such that

$$|\langle R(\exp tH)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}| \leq Ce^{t(s_{H_0}-2\rho(H))}$$

for all $t \geq 0$ and $H \in B_\delta(H_0)$. By shrinking δ we can assume that $s_{H_0} < (2-2(p+\varepsilon)^{-1})\rho(H)$ for any $H \in \overline{B}_\delta(H_0)$. By compactness of the unit sphere in $\mathfrak{a}$ we only need finitely many H_0^i in order to have

$$\overline{\mathfrak{a}_+} \subseteq \bigcup_i \mathbb{R}_+ \cdot \tilde{B}_i \text{ where } \tilde{B}_i := B_{\delta_i}(H_0^i) \cap \{H \in \mathfrak{a}, \|H\| = 1\}.$$

Therefore using the right K-invariance of f_i and the integral formula (2.1) for the decomposition $G = K\exp(\overline{\mathfrak{a}_+})K$ we get,

$$\begin{aligned}\int_G |\langle R(h)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}|^{p+\varepsilon}\,dh &= \int_{\mathfrak{a}_+} |\langle R(\exp H)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}|^{p+\varepsilon}\delta(H)\,dH \\ &\leq \sum_i \int_{\mathbb{R}_+\tilde{B}_i} |\langle R(\exp H)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}|^{p+\varepsilon}\delta(H)\,dH.\end{aligned}$$

Hence, it suffices to show for the finitely many i that

$$\int_{\mathbb{R}_+\tilde{B}_i} |\langle R(\exp H)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}|^{p+\varepsilon}\delta(H)\,dH \tag{4.2}$$

is finite. Using polar coordinates and $\delta(H) \leq e^{2\phi(H)}$ (4.2) is bounded by

$$C' \sup_{H\in\tilde{B}_i} \int_0^\infty |\langle R(\exp tH)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}|^{p+\varepsilon} e^{2\rho(tH)} t^{\dim(\mathfrak{a})-1}\,dt.$$

Now we use Theorem 10 to obtain that (4.2) is bounded by

$$CC' \sup_{H\in B_\delta(H_0)\cap\{\|\cdot\|=1\}} \int_0^\infty e^{t((s_{H_0}-2\rho(H))(p+\varepsilon)+2\rho(H))} t^{\dim(\mathfrak{a})-1}\,dt$$

which is finite since $s_{H_0} < \rho(H)(2-2(p+\varepsilon)^{-1})$ for any $H \in \overline{B}_\delta(H_0)$ normalized. This completes the proof of Theorem 11. □

Before proving Theorem 10 let us prove the following lemma that is certainly known to experts but might still be of independent interest.

Recall (see [Hel84, Prop. I.5.21]) that the mapping

$$(\overline{n}, m, a, n) \mapsto \overline{n}man \in G$$

is a bijection of $\overline{N} \times M \times A \times N$ onto an open submanifold of G whose complement has Haar measure 0. Moreover,

$$\int_G f(g)\,dg = \int_{\overline{N}\times M\times A\times N} f(\overline{n}man)e^{2\rho(\log a)}\,d\overline{n}\,dm\,da\,dn.$$

Lemma 12. *Let $\varphi_1, \varphi_2 \in C_c(G)$ with $\operatorname{supp}\varphi_i \subseteq \overline{N}MAN$. Then there is a constant $C = C_{\varphi_1,\varphi_2}$ such that for all $h \in A$*

$$\left|\int_G \varphi_1(h^{-1}gh)\varphi_2(g)\,dg\right| \leq Ce^{-2\rho(\log h)}.$$

Proof. By the triangle inequality we can assume that $\varphi_i \geq 0$. Since $\operatorname{supp} \varphi_i \subseteq \overline{N}MAN$ there exist compact sets $C_{\overline{N}} \subseteq \overline{N}$, $C_A \subseteq A$, and $C_N \subseteq N$ with $\operatorname{supp} \varphi_i \subseteq C_{\overline{N}} M C_A C_N$. We thus have

$$\begin{aligned} c := c_{\varphi_1,\varphi_2,h} &:= \int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg \\ &= \int_{C_{\overline{N}}\times M\times C_A\times C_N} \varphi_1(h^{-1}\overline{n}manh)\varphi_2(\overline{n}man)e^{2\rho(\log a)}\, d\overline{n}\, dm\, da\, dn \\ &\leq \|\varphi_2\|_\infty \int_{C_{\overline{N}}\times M\times C_A\times C_N} \varphi_1(h^{-1}\overline{n}manh)e^{2\rho(\log a)}\, d\overline{n}\, dm\, da\, dn. \end{aligned}$$

Since M centralizes A and A is abelian

$$c \leq \|\varphi_2\|_\infty \int_{C_{\overline{N}}\times M\times C_A\times C_N} \varphi_1(h^{-1}\overline{n}hmah^{-1}nh)e^{2\rho(\log a)}\, d\overline{n}\, dm\, da\, dn.$$

Estimating φ_1 by its absolute value and using that A normalizes both N and $\overline{N}$ we get

$$\begin{aligned} c &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_M dm \int_{C_A} e^{2\rho(\log a)}\, da \int_{C_{\overline{N}}\cap hC_{\overline{N}}h^{-1}} d\overline{n} \int_{C_N\cap hC_Nh^{-1}} dn \\ &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_M dm \int_{C_A} e^{2\rho(\log a)}\, da \int_{C_N} dn \int_{hC_{\overline{N}}h^{-1}} d\overline{n}. \end{aligned}$$

Since the Jacobian factor for the diffeomorphism $\overline{n} \mapsto h^{-1}\overline{n}h$ of $\overline{N}$ is $\det \operatorname{Ad}(h)|_{\overline{\mathfrak{n}}} = e^{-2\rho(\log h)}$ we have

$$\int_{hC_{\overline{N}}h^{-1}} d\overline{n} = \int_{\overline{N}} 1_{C_{\overline{N}}}(h^{-1}\overline{n}h)\, d\overline{n} = \int_{\overline{N}} 1_{C_{\overline{N}}}(\overline{n})e^{-2\rho(\log h)}\, d\overline{n} = \int_{C_{\overline{N}}} d\overline{n}\, e^{-2\rho(\log h)}.$$

We conclude

$$c_{\varphi_1,\varphi_2,h} \leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_M dm \int_{C_A} e^{2\rho(\log a)}\, da \int_{C_N} dn \int_{C_{\overline{N}}} d\overline{n}\, e^{-2\rho(\log h)} = C_{\varphi_1,\varphi_2}e^{-2\rho(\log h)}$$

proving the theorem. □

Let us now prove Theorem 10.

Proof of Theorem 10. Let $f_1, f_2 \in C_c(\Gamma\backslash G)$. We can find $\tilde{f}_i \in C_c(G)$ such that $f_i(\Gamma g) = \sum_{\gamma\in\Gamma} \tilde{f}_i(\gamma g)$.

We then have

$$\begin{aligned} \langle R(h)f_1, f_2\rangle_{L^2(\Gamma\backslash G)} &= \int_{\Gamma\backslash G} f_1(\Gamma gh)f_2(\Gamma g)\, d\Gamma g = \int_G \tilde{f}_1(gh)f_2(\Gamma g)\, dg \\ &= \sum_{\gamma\in\Gamma} \int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg. \end{aligned} \tag{4.3}$$

For any $g \in G$ there is an open neighborhood U_g of g such that $U_g^{-1}U_g \subseteq \overline{N}MAN$ since $\overline{N}MAN$ is an open neighborhood of the identity element. Since $\operatorname{supp} \tilde{f}_i$ is compact there are finitely many g_k such that $\operatorname{supp} \tilde{f}_i \subseteq \bigcup U_{g_k}$. There exists a partition of unity χ_k subordinate to U_{g_k}, i.e. $\chi_k \in C_c(G)$ with $\operatorname{supp} \chi_k \subseteq U_{g_k}$ and $\sum_k \chi_k(x) = 1$ for all $x \in \operatorname{supp} \tilde{f}_i$. We decompose $\tilde{f}_i$ as $\sum_k \chi_k \tilde{f}_i$ in (4.3). This allows us to assume without loss of generality that $\operatorname{supp} \tilde{f}_i$ is contained in some U_g, since we can estimate each of the finite summands individually. In particular, we can assume that $(\operatorname{supp} \tilde{f}_i)^{-1} \operatorname{supp} \tilde{f}_i \subseteq \overline{N}MAN$.

Let $\gamma \in \Gamma$ such that $\int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg \neq 0$. Then there is $g \in G$ with $gh \in \operatorname{supp} \tilde{f}_1$ and $\gamma g \in \operatorname{supp} \tilde{f}_2$. Therefore, $\gamma \in (\operatorname{supp} \tilde{f}_2)g^{-1} \subseteq \operatorname{supp} \tilde{f}_2 h(\operatorname{supp} \tilde{f}_1)^{-1}$. Hence, there are s_1 and s_2 in $\operatorname{supp} \tilde{f}_1$ and $\operatorname{supp} \tilde{f}_2$, respectively, with $\gamma = s_2 h s_1^{-1}$. By change of variables

$$\begin{aligned}\int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg &= \int_G \tilde{f}_1(gh)\tilde{f}_2(s_2 h s_1^{-1} g)\, dg = \int_G \tilde{f}_1((hs_1^{-1})^{-1}gh)\tilde{f}_2(s_2 g)\, dg \\ &= \int_G \tilde{f}_1(s_1 h^{-1} gh)\tilde{f}_2(s_2 g)\, dg.\end{aligned}$$

If we define $\varphi_i(g) := \max_{s\in\operatorname{supp} \tilde{f}_i} |\tilde{f}_i(sg)|$ we can estimate

$$\left|\int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg\right| \leq \int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg.$$

Hence we have

$$|\langle R(h)f_1, f_2\rangle| \leq \#(\Gamma \cap (\operatorname{supp} \tilde{f}_2)h(\operatorname{supp} \tilde{f}_1)^{-1}) \int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg.$$

Note that if $\varphi_i(g) \neq 0$ then there is $s \in \operatorname{supp} \tilde{f}_i$ such that $sg \in \operatorname{supp} \tilde{f}_i$. Hence, $\operatorname{supp} \varphi_i \subseteq (\operatorname{supp} \tilde{f}_i)^{-1} \operatorname{supp} \tilde{f}_i$ is compact and contained in $\overline{N}MAN$. Therefore, by Lemma 12

$$\int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg \leq Ce^{-2\rho(\log h)}.$$

The theorem now follows from Lemma 13 and Lemma 14 below. ☐

Lemma 13 (see [Ben96, Prop. 5.1])**.** *For all compact sets $C \subseteq G$ there exists a compact set $L \subseteq \mathfrak{a}$ such that $\mu(CgC) \subseteq \mu(g) + L$.*

Lemma 14. *For all $H_0 \in \mathfrak{a}_+$ normalized, all $L \subseteq \mathfrak{a}$ compact, all t large enough, and all $s > \psi_\Gamma(H_0)$ with $s \geq 0$ there exists $\delta > 0$ and $C > 0$ such that*

$$\#\{\gamma \in \Gamma \mid \mu(\gamma) \in tH + L\} \leq Ce^{ts}$$

for $H \in B_\delta(H_0)$ normalized.

Remark. If $\psi_\Gamma(H_0) < 0$ then H_0 is not in the limit cone and $\psi_\Gamma(H_0) = -\infty$. Moreover, there is an open cone containing H_0 that contains only finitely many Γ points. In particular, $\{\gamma \in \Gamma \mid \mu(\gamma) \in tH + L\}$ is empty for t large enough depending on H_0.

Proof. By definition there exists an open cone $\mathcal{C}$ containing H_0 such that

$$\sum_{\gamma\in\Gamma, \mu(\gamma)\in\mathcal{C}} e^{-s\|\mu(\gamma)\|} < \infty.$$

Therefore for any $R > 0$, there is $C > 0$ such that

$$\#\{\gamma \mid \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \in]t - R, t + R]\} \leq Ce^{ts}.$$

Note that for every $\delta > 0$ with $\overline{B_\delta(H_0)} \subseteq \mathcal{C}$ there is $t_0 > 0$ such that $tH + L \subseteq \mathcal{C}$ for every $t \geq t_0$ and $H \in B_\delta(H_0)$. If we take $R > 0$ is such that $L \subseteq B_R(0)$ then we can estimate for all $t \geq t_0$ and $H \in B_\delta(H_0)$, normalized

$$\begin{aligned}\#\{\gamma \mid \mu(\gamma) \in tH + L\} &\leq \#\{\gamma \mid \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \in]t\|H\| - R, t\|H\| + R]\} \\ &\leq Ce^{ts}.\end{aligned}$$

☐

4.1. **Proof of Theorem 2.**

Proof of Theorem 2. (i) and (ii) are equivalent by Lemma 7 and 9. (ii) implies (iii) by Lemma 6. (iii) implies (iv) by Theorem 11. If $p \in 2\mathbb{N}$, (iv) implies (ii) by Theorem 5. (iii) implies (v) by [WZ23, Cor. 1.4]:

$$\begin{aligned}\inf \sigma(\Delta) &= \|\rho\|^2 - \max\left\{0, \sup_{H\in\overline{\mathfrak{a}_+}} \frac{\psi_\Gamma(H) - \langle \rho, H\rangle}{\|H\|}\right\}^2 \\ &\geq \|\rho\|^2 - (1-2p^{-1})^2 \left(\sup_{H\in\overline{\mathfrak{a}_+}} \frac{\langle \rho, H\rangle}{\|H\|}\right)^2 \\ &= (1-(1-2p^{-1})^2)\|\rho\|^2 = 2p^{-1}(2-2p^{-1})\|\rho\|^2.\end{aligned}$$

□

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