An Abstract Spectral Approach to Horospherical Equidistribution

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Abstract

This paper introduces an abstract spectral approach to prove effective equidistribution of expanding horospheres in hyperbolic manifolds. The method, which is motivated by the approach to counting developed by (Lax-Phillips 1982), produces highly effective, explicit error terms. To exhibit the flexibility of this method we prove effective horospherical equidistribution theorems in $T^1(\mathbb{H}^{n+1})$ and in the higher rank setting, $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$.

1 Introduction

Given a manifold, a flow, and an expanding submanifold under the action of the flow, a key question in dynamical systems is to establish when such a submanifold equidistributes, and moreover, to establish effective rates of equidistribution. Establishing this behavior is at the heart of some of the crowning achievements of homogeneous dynamics (e.g Ratner's theorems [Rat91a, Rat91b], and horospherical equidistribution [Zag81, Sar81, OS13]). Moreover, these equidistribution results are crucial to the application of dynamical methods to number theory problems. One such dynamical method, uses the Margulis thickening trick to pass from mixing to equidistribution statements.

While this dynamical method is very strong, it is in some sense wasteful. That is, there are some more specific questions for which spectral methods have proven to be more effective and flexible. In particular, for counting points in group orbits an abstract spectral method developed by Lax and Phillips [LP82] (itself modeled on previous approaches of Huber, Patterson, and Selberg [Hub56, Pat75, Sel14]) was used to obtain effective asymptotic results beyond the reach of these dynamical equidistribution methods. Moreover, the abstract spectral approach is extremely 'soft', in that it requires very little specific input from the problem. Thus, the method is potentially extremely flexible (see [Kon09, KL22] for extensions).

In this paper, we introduce a method for proving effective horospherical equidistribution which is based on the abstract spectral method to counting. To exhibit this method we will recover and improve on two existing effective horospherical equidistribution theorems. In the rank one setting we recover (with an improved error rate) an effective equidistribution theorem of Mohammadi and Oh [MO15] for expanding horospheres in $\mathrm{Isom}^+(\mathbb{H}^{n+1})$. Then we extend our method to higher rank groups in order to prove an effective equidistribution theorem for expanding horospheres in quotients of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. This relates to numerous papers, for example [EMV09, KM12, MG14, Yan16, AELM20], and several others.

Remark. It is worth noting that a related abstract spectral method was used by Strömbergsson [Str13], Södergren [Sï2], and Edwards [Edw17, Edw21] to tackle the same types of problems. In fact in some contexts their method achieves stronger error terms. That is, because they can avoid smoothing, their T dependence (the height of the horosphere) is improved, but they sacrifice in the dependence on the regularity of the test function. Since, in application, the test functions are normally approximations to indicator functions, this dependence can be very important. In addition, the method developed herein allows us to tackle somewhat more general problems: in the higher rank case we can expand in different directions with different rates.

1.1 Plan of paper

In the remainder of the introduction, we present our main theorems for $T^1(\mathbb{H}^{n+1})$, $\mathrm{SL}_3(\mathbb{R})$, and $\mathrm{SL}_n(\mathbb{R})$ in that order. The rest of the paper is arranged as follows

- Section 2 presents some preliminaries on rank one hyperbolic geometry and spectral theory.

- Section 3 presents a simplified version the proof of Theorem 1 in the special case when $n = 1$ and the function is right K-invariant. This acts as a warm-up.

- Section 4 gives the full proof of Theorem 1 for general n. That is, the rank one setting.

- Section 5 presents the necessary preliminaries in the higher rank setting.

- Section 6 presents the proof of Theorem 2, the $\mathrm{SL}_3(\mathbb{R})$ result.

- Section 7 then presents the proof of Theorem 3, the $\mathrm{SL}_n(\mathbb{R})$ result.

Where possible we will avoid repeating details.

1.2 Main theorem – rank 1

Let $n \geq 1$ be fixed, and consider the $n + 1$-dimensional hyperbolic upper half-space

$$\mathbb{H}^{n+1} := \{(x_1, \ldots, x_n, y) \; : \; y > 0\}.$$

Let $G := \mathrm{Isom}^+(\mathbb{H}^{n+1})$, the group of orientation preserving isometries; which is isomorphic to the identity component of $\mathrm{SO}(n + 1, 1)$. When $n = 1, 2$, G is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$ respectively. Fix a discrete, not co-compact, Zariski dense, geometrically finite subgroup $\Gamma < G$. Moreover assume the critical exponent $\delta > n/2$.

Consider the right regular representation of G acting on $L^2(\Gamma \backslash G)$ (see Section 2 for definitions). We will also assume that the right regular representation of G acting on the space $L^2(\Gamma \backslash G)$ has no complementary series representation. We call such a subgroup Γ 'nice'. For $n = 1$ and $n = 2$ all of the groups we consider are nice. For $n > 2$ this is quite a restriction, but we impose it to avoid having to dismiss terms coming from these non-spherical complementary series representations. Recently, Edwards-Oh have treated the contribution of these non-spherical complementary series representations to a related problem in some detail [EO21], however it is simpler for us to restrict our results to nice subgroups.

Fix a Cartan decomposition of $G = HAK$ where K is a maximal compact subgroup, $A := \{a_y \; : \; y \in \mathbb{R}_{>0}\}$ is a one parameter diagonalizable subgroup, and $H := \{g \in G \; : \; a_y g a_{1/y} \to e$ as $y \to 0\}$ is the expanding horospherical subgroup for $a_{1/y}$. Further, let M be the centralizer of A in K. Given an $F \in C_c^\infty(\Gamma \backslash G / M)$, our goal in this paper is to study the asymptotic behavior of the average

$$\mathcal{M}_T(F) := \int_{h \in \Gamma_H \backslash H} F(h a_{1/T}) \mathrm{d}h, \tag{1.1}$$

in the limit as $T \to \infty$. Here $\Gamma_H := \Gamma \cap H$ and $\mathrm{d}h$ denotes the Haar measure on H.

To state our main theorem let

$$-\Delta := y^2(\partial_{x_1 x_1} + \cdots + \partial_{x_n x_n} + \partial_{yy}) - (n - 1)y\partial_y$$

denote the hyperbolic Laplacian. Then, from work of Lax-Phillips [LP82], the $L^2(\Gamma \backslash \mathbb{H}^{n+1})$ spectrum of the Laplacian consists of finitely many discrete eigenvalues $\lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k$ in the region $[0, (n/2)^2)$ and continuous spectrum above $n^2/4$. Write $\lambda_j = s_j(n - s_j)$ with $n/2 \leq s_j \leq \delta$. Then by work of Patterson [Pat76], and Sulliven [Sul84], we have $s_0 = \delta$.

Finally, let $\|F\|_{1,\infty} := \|\partial_y F(x + iy)\|_\infty$ denote the first Sobolev norm of F, where $\|\cdot\|_\infty$ is the L^∞ norm on $\Gamma \backslash G$. Throughout we let $\|F\|_\Gamma = \|F\|_{L^2(\Gamma \backslash G)}$. To ease notation, let $P(n) := n^2 - 3n + 10$. Our main theorem, in rank one, is the following

Theorem 1. *Let $\Gamma < G$ be a 'nice', geometrically finite, Zariski dense subgroup with critical exponent $\delta > n/2$. Let H be a horospherical subgroup. Then for any $F \in C_c^\infty(\Gamma \backslash G / M)$, there exists a constant c_0, such that*

$$\mathcal{M}_T(F) = c_0 T^{\delta-n}(1 + O(T^{-\eta_{cont}}(\log T)^{\frac{4}{P(n)}} + T^{-\eta_{s_1}})\|F\|_\Gamma^{4/P(n)}\|F\|_{1,\infty}^{\frac{P(n)-4}{P(n)}}) \qquad (1.2)$$

where

$$\eta_{cont} = \frac{4(\delta - n/2)}{P(n)} \qquad and \qquad \eta_{s_1} = \frac{4(\delta - s_1)}{P(n)}.$$

Further, if F is right K-invariant, there exist constants $c_0, c_1 \dots, c_k$ such that

$$\mathcal{M}_T(F) = c_0 T^{\delta-n} + c_1 T^{s_1-n} + \cdots + c_k T^{s_k-n} + O(T^{\delta/3-2n/3}(\log T)^{2/3}\|F\|_\Gamma^{2/3}\|F\|_{1,\infty}^{1/3}), \qquad (1.3)$$

(1.3) *holds even if Γ is not nice.*

Remark. This theorem improves on [MO15, Theorem 1.7] in that the error exponent η is sharper. While [MO15] do not state an explicit error term, a similar method is used in [LO13] in the case $n = 1, 2$. For $n = 2$ their $\eta_{s_1} = 2(\delta - s_1)/7$ while ours is $(\delta - s_1)/2$. Moreover Oh-Shah [OS13] show that the coefficient c_0 is proportional to $\mathrm{m}^{BR}(F)$ – the Burger-Roblin mass of F. However, while [Edw17] achieves a stronger error term in the T aspect, Theorem 1 is stronger in the Sobolev norm of F. Thus, for example when counting circles in an Apollonian circle packing, Edwards achieves an error term of size $T^{\delta-(\delta-s_1)/9}$ while applying Theorem 1 achieves an error term of size $T^{\delta-2(\delta-s_1)/9}$ (although both results are worse than the more direct method of Kontorovich and the author [KL22]).

Remark. Note that for K-invariant functions we can extract all lower order terms coming from the discrete spectrum. This is a consequence of the fact that the subspace of K-fixed vectors in the jth eigenspace is one dimensional, while the full j^{th} eigenspace is infinite dimensional, which prevents us from extracting some dependencies.

Remark. One could possibly use the machinery of Edwards-Oh [EO21] to remove the condition that Γ is nice. We leave this extension for future work.

1.3 Main theorem − $\mathrm{SL}_3(\mathbb{R})$

To keep matters simple, for higher rank, we first present our results on $\mathrm{SL}_3(\mathbb{R})$. For this subsection, let $G := \mathrm{SL}_3(\mathbb{R})$. Consider the following subgroups

- $A := \left\{ a(\mathbf{y}) := \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mathbf{y} \in \mathbb{R}_{>0}^2 \right\}$ the two-parameter diagonal subgroup.

- $H := \left\{ h(\mathbf{x}) := \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} : \mathbf{x} \in \mathbb{R}^3 \right\}$ the upper-triangular subgroup.

- $K = \mathrm{SO}(3, \mathbb{R})$ the maximal compact subgroup of G.

Then $\mathrm{GL}_3(\mathbb{R}) = HAKZ_3$ where Z_3 is a one parameter diagonal subgroup. Let $\mathcal{H} := G/KZ_3 \cong HA$ denote the generalized upper half plane (see [Gol06]) or the space of lattices. The natural, invariant metric on $\mathcal{H}$ is

$$\mathrm{d}V = \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}y_1 \mathrm{d}y_2}{y_1^3 y_2^3}$$

There are two Casimir operators in this case

$$-\Delta_1 := y_1^2 \partial_{y_1 y_1} + y_2^2 \partial_{y_2 y_2} - y_1 y_2 \partial_{y_1 y_2} + y_1^2 (x_2^2 + y_2^2) \partial_{x_3 x_3} + y_1^2 \partial_{x_1 x_1} + y_2^2 \partial_{x_2 x_2} + 2 y_1 x_2 \partial_{x_1 x_3},$$

$$\begin{aligned}
-\Delta_2 := & -y_1^2 y_2 \partial_{y_1 y_1 y_2} + y_1 y_2^2 \partial_{y_1 y_2 y_2} - y_2^2 \partial_{y_2 y_2} + y_1^2 \partial_{y_1 y_1} \\
& - y_1^3 y_2^2 \partial_{x_3 x_3 y_1} + y_1 y_2^2 \partial_{x_2 x_2 y_1} - 2 y_1^2 y_2 x_2 \partial_{x_1 x_3 y_2} + (-x_2 + y_2) y_1^2 y_2 \partial_{x_3 x_3 y_2} \\
& - y_1^2 y_2 \partial_{x_1 x_1 y_2} + 2 y_1^2 y_2^2 \partial_{x_1 x_2 x_3} + 2 y_1^2 y_2 x_2 \partial_{x_2 x_3 x_3} + 2 y_1^2 x_2 \partial_{x_1 x_3} + y_1^2 (x_2^2 + y_2^2) \partial_{x_3 x_3} \\
& + y_1^2 \partial_{x_1 x_1} - y_2^2 \partial_{x_2 x_2}.
\end{aligned}$$

Let $\Gamma < \mathrm{SL}_3(\mathbb{Z})$ be a co-finite subgroup. Consider the Hilbert space $\mathscr{H} = L^2(\Gamma \backslash G)$, G acts by the right regular representation on $\mathscr{H}$, which decomposes into irreducible subspaces (see Section 5). Both Δ_1 and Δ_2 act as scalars on each irreducible subspace. Thus, up to multiplicity we can parameterize the irreducible subspace by a pair (λ_1, λ_2). If we consider the spectrum of Δ_1, then this consists of finitely many eigenvalues

$$0 = \lambda_1^{(0)} < \lambda_1^{(1)} < \cdots < \lambda_1^{(k_1)} < 1$$

of finite multiplicity. With continuous spectrum above 1. Since we work with co-finite subgroups, the base eigenvalue is 0, and since the group satisfies property T, we know there is a spectral gap. In particular for $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ we have that $\lambda_1^{(1)} > 1$ (see [Mil02]). Let $\boldsymbol{\lambda}^{(i)}$ for $i = 0, \ldots, k$ denote the discrete points in the joint spectrum of λ_1 and λ_2. Any point in the discrete spectrum has finite multiplicity.

Rather than work with $\boldsymbol{\lambda}^{(i)}$ it is more practical to work with vectors $\boldsymbol{\nu} = (\nu_1, \nu_2) \in [0, 1)^2$. Let (ν_1, ν_2) solve the equations $\lambda_1 = -3(\nu_1^2 + \nu_2^2 - \nu_1 - \nu_2 + \nu_1 \nu_2$ coming from Δ_1 and $\lambda_2 = (\nu_2 - \nu_1)(2\nu_1^2 + \nu_2(-3 + 2\nu_2) + \nu_1(-3 + 5\nu_2))$, coming from Δ_2. These equations come from eigenvalue equations for the I-function $I(z) = y_1^{\nu_1 + 2\nu_2} y_2^{2\nu_1 + \nu_2}$ and there are 6 solutions to the pair of equations, call them $(\nu_{i,1}, \nu_{i,2})$ for $i = 1, \ldots, 6$. The continuous spectrum corresponds to $\nu_1 = 1/3 + it_1$ and $\nu_2 = 1/3 + it_2$. For convenience we also define

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$s = \nu_1 + 2\nu_2$ and $r = \nu_2 + 2\nu_1$ (the exponents of y_1 and y_2 respectively, in the I function).

For $F \in C_c^\infty(\Gamma \backslash \mathcal{H})$, our aim is to study the asymptotic behavior of

$$\mathcal{M}_{\mathbf{T}}(F) := \int_{\Gamma_H \backslash H} F(ha(1/T_1, 1/T_2))\mathrm{d}h.$$

where $\mathbf{T} = (T_1, T_2)$. Now, for any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ define the following quantity

$$m_{\mathbf{T}}(\boldsymbol{\lambda}) := \sum_{i=1}^6 c_i T_1^{s_i} T_2^{r_i}, \tag{1.4}$$

for some constants c_i which could be made explicit. We need this cumbersome quantity since the term governing the error term will depend on the (a priori unknown) relationship between s, r, T_1 and T_2.

Theorem 2. *Let $F \in C_c^\infty(\Gamma \backslash \mathcal{H})$, then there exist constants $c_0, c_1, \ldots, c_k < \infty$, depending only on Γ and F, such that*

$$\mathcal{M}_{\mathbf{T}}(F) = c_0 m_{\mathbf{T}}(0) + c_1 m_{\mathbf{T}}(\boldsymbol{\lambda}^{(1)}) + \cdots + c_k m_{\mathbf{T}}(\boldsymbol{\lambda}^{(k)}) \tag{1.5}$$
$$+ O(T_1^{-1/2} T_2^{-1/2}(\|F\|_{1,\infty} \|F\|_{\Gamma} \log T_1 \log T_2)^{1/2}),$$

in the limit as $T_1 T_2 \to \infty$. In particular, if $T_1, T_2 \to \infty$ the main term is $C \int_{\Gamma \backslash \mathcal{H}} F(z)\mathrm{d}z$.

In the higher rank we write $\|F\|_{1,\infty} := \|\nabla_{\mathbf{y}} F(\mathbf{x}, \mathbf{y})\|_\infty$.

Remark. A similar proof method was used by Edwards [Edw21] to achieve an excellent error term in the more general Lie group setting. Theorem 2 is still worthwhile proving, given the simplicity and directness of our method here, and since the F dependence is extremely good. Thus, while Edwards achieves a better T dependence, he sacrifices in the F dependence.

1.4   Main theorem $-$ $\mathrm{SL}_n(\mathbb{R})$

Turning now to the group $G := \mathrm{SL}_n(\mathbb{R})$ for $n \geq 2$. Again, we employ an Iwasawa decomposition. Consider the following subgroups

- $A := \left\{ a(\mathbf{y}) := \begin{pmatrix} y_1 \cdots y_{n-1} & & & & \\ & y_1 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} : \mathbf{y} \in \mathbb{R}_{>0}^{n-1} \right\}$ the $n-1$-parameter diagonal subgroup.

- $H := \left\{ h(\mathbf{x}) := \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & \cdots & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix} : x_i \in \mathbb{R} \right\}$ the upper-triangular sub-

group of dimension $n(n-1)/2$.

- $K = \mathrm{SO}_n(\mathbb{R})$ the maximal compact subgroup of G.

Once again we can write $\mathrm{GL}_n(\mathbb{R}) = HAKZ_d$ with Z_d a one-parameter subgroup. Let $\mathcal{H} := HA$ denote the generalized upper half-space. The invariant metric on $\mathcal{H}$ is

$$\mathrm{d}V = \prod_{1 \leq i < j \leq n} \mathrm{d}x_{i,j} \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} \mathrm{d}y_k.$$

Let $\Delta_1, \ldots, \Delta_{n-1}$ denote the Casimir operators.

Let $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ be a cofinite subgroup with $\Gamma_H \backslash G$ is closed. Again, the Casimir operators act like scalars on the space of irreducibles. We denote the exceptional values (that is the values outside the tempered spectrum)

$$0 = \boldsymbol{\lambda}^{(0)} < \boldsymbol{\lambda}^{(1)} \leq \cdots \leq \boldsymbol{\lambda}^{(k)},$$

where $\lambda_i^{(j)}$ is the value of Δ_i acting on that irreducible subspace.

Rather than work with the spectral parameters $\boldsymbol{\lambda}$ it is more convenient to work with the corresponding $\boldsymbol{\nu}$ coming from the I-functions. That is, set

$$I_{\boldsymbol{\nu}}(\mathbf{y}) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij}\nu_j}$$

where

$$b_{ij} := \begin{cases} ij, & \text{if } i + j \leq n, \\ (n-i)(n-j), & \text{if } i + j \geq n. \end{cases}$$

Furthermore, let $s_i := \sum_{j=1}^{n-1} b_{ij}\nu_j$ denote the exponent of y_i in the I function. Then the $\boldsymbol{\nu}$ solve the system of equations

$$\Delta_k I_{\boldsymbol{\nu}}(\mathbf{y}) = \lambda_k \Delta_k I_{\boldsymbol{\nu}}(\mathbf{y}), \qquad \text{for all } k = 1, \ldots, n-1$$

for a given point in the spectrum there are $L = n!$ such solutions, denoted $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_L$. For the discrete points in the exceptional spectrum (which are expected not to exist) $\boldsymbol{\lambda}^{(i)}$ we denote the associated $\boldsymbol{\nu}_j$ by $\boldsymbol{\nu}_j^{(i)}$. The exceptional spectrum lies in $[0, 1/n]^{n-1}$

[Gol06] and the continuous spectrum corresponds to points $\boldsymbol{\nu}$ with $\nu_j = 1/n + it_j$ for each $j = 1, \ldots, n-1$. For convenience we define the following quantity

$$m_{\mathbf{T}}(\boldsymbol{\lambda}) = \sum_{j=1}^{L} I_{\boldsymbol{\nu}_j}(\mathbf{y}).$$

Our goal is to understand the asymptotic behavior of the following horospherical average, let $F \in C_c^\infty(\Gamma \backslash \mathcal{H})$ and let $\mathbf{T} \in \mathbb{R}_{>0}^{n-1}$

$$\mathcal{M}_{\mathbf{T}}(F) := \int_{\Gamma_H \backslash H} F(ha(1/\mathbf{T})) \mathrm{d}h.$$

For that purpose, let $I_{cont}(\mathbf{T}) := \prod_{i=1}^{n-1} T_i^{(b_{i1}+\cdots+b_{i(n-1)})/n}$. The equidistribution theorem for $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n, \mathbb{R})$ is the following

Theorem 3. *Suppose Γ is as above. Let $F \in C_c^\infty(\Gamma \backslash \mathcal{H})$, then there exist constants $c_0, c_1, \ldots, c_k < \infty$, depending only on Γ and F, such that*

$$\mathcal{M}_{\mathbf{T}}(F) = c_0 m_{\mathbf{T}}(0) + c_1 m_{\mathbf{T}}(\boldsymbol{\lambda}^{(1)}) + \cdots + c_k m_{\mathbf{T}}(\boldsymbol{\lambda}^{(k)}) \tag{1.6}$$
$$+ O\left((I_{cont}(\mathbf{T}) \log T_1 \cdots \log T_{n-1})^{\frac{2}{n+1}} \|F\|_\Gamma^{\frac{2}{n+1}} \|F\|_{1,\infty}^{\frac{n-1}{n+1}} \right),$$

in the limit as $\prod_{i=1}^{n-1} T_i \to \infty$. In particular, if all $T_i \to \infty$ the main term is $C \int_{\Gamma \backslash \mathcal{H}} F(z) \mathrm{d}z$ for some $C > 0$.

2 Preliminaries and Notation – Rank One

Given an $n \geq 1$ we work in the group $G := \mathrm{Isom}(\mathbb{H}^{n+1})$. Throughout, let $\langle \cdot, \cdot \rangle_\Gamma$ denote the $L^2(\Gamma \backslash G)$ inner product with respect to the Haar measure $\mathrm{d}g$. Let $\mathfrak{g} := \mathrm{Lie}(G)$, note that $d := \dim(\mathfrak{g}) = (n+1)(n+2)/2$. We have the following subgroups

- $A := \{a_y \ : \ y > 0\}$ - a one dimensional diagonalizable subgroup.

- $H := \{g \in G \ : \ a_y g a_{1/y} \to e \text{ as } y \to 0\}$ - the expanding horospherical subgroup associated to A – $\dim(H) = n$.

- K - the maximal compact subgroup – $\dim(K) = (n+1)n/2$.

- M - the centralizer of A in K – $\dim(M) = n - 1$.

Further, let $\overline{K} := K/M$. Note that $\mathbb{H}^{n+1} = G/K$ and $T^1(\mathbb{H}^{n+1}) = G/M$. Throughout, we use Γ_X to denote $\Gamma \cap X$.

Lie algebras and the Casimir Operator: In general, the Casimir operators generate the center of the universal enveloping algebra. The elements of the Lie algebra act like first order differential operators, and the Casimir operator acts as a second order differential operator on smooth functions on G. When restricted to right K-invariant smooth functions on G, the Casimir operator $\mathcal{C}$ agrees with the hyperbolic Laplacian Δ.

Decomposition of $L^2(\Gamma\backslash G)$ into irreducibles: The group G acts by the right-regular representation on the Hilbert space $\mathscr{H} := L^2(\Gamma\backslash G)$ of square-integrable Γ-automorphic functions. The space $\mathscr{H}$ splits into components as follows:

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k \oplus \mathscr{H}^{tempered}. \tag{2.1}$$

Each of the subspaces $\mathscr{H}_j$ denotes the G-span of the eigenfunction with eigenvalue λ_j. $\mathscr{H}^{tempered}$ denotes the tempered spectrum. Since we work with nice subgroups, there is no nonspherical complimentary series. For $n \geq 2$ the space $\mathscr{H}_j$ is infinite dimensional. However the subspace of K-fixed vectors in $\mathscr{H}_j$ is one dimensional.

2.1 Abstract Spectral Theorem:

The abstract spectral theorem is a remarkable theorem coming from abstract operator theory (see for example [Rud73, Ch. 13]). Since it represents the crucial input into our method we state it in full generality here. Let L be a self-adjoint, positive semidefinite operator on the Hilbert space $\mathscr{H}$. In our applications $\mathscr{H}$ will be $L^2(\Gamma\backslash G)$ and L will be the Casimir operator (or Laplacian).

Theorem 4 (Abstract Spectral Theorem). *There exists a spectral measure $\widetilde{\mu}$ on $\mathbb{R}$ and a unitary spectral operator $\widehat{} : \mathscr{H} \to L^2([0,\infty), d\widetilde{\mu})$ such that:*

i) Abstract Parseval's Identity: for $\phi_1, \phi_2 \in \mathscr{H}$

$$\langle \phi_1, \phi_2 \rangle_{\mathscr{H}} = \langle \widehat{\phi}_1, \widehat{\phi}_2 \rangle_{L^2([0,\infty), d\widetilde{\mu})}; \tag{2.2}$$

ii) The spectral operator is diagonal with respect to L: for $\phi \in \mathscr{H}$ and $\lambda \geq 0$,

$$\widehat{L\phi}(\lambda) = \lambda\widehat{\phi}(\lambda); \tag{2.3}$$

If λ is in the point specturm of L with associated eigenspace $\mathscr{H}_\lambda$, then for any $\psi_1, \psi_2 \in \mathscr{H}$ one has

$$\widehat{\psi_1}(\lambda)\overline{\widehat{\psi_2}(\lambda)} = \langle \mathrm{Proj}_{\mathscr{H}_\lambda} \psi_1, \mathrm{Proj}_{\mathscr{H}_\lambda} \psi_2 \rangle, \tag{2.4}$$

where Proj refers to the projection to the subspace $\mathscr{H}_\lambda$. In the special case that $\mathscr{H}_\lambda$ is one-dimensional and spanned by the normalized eigenfunction ϕ_λ, we have that

$$\widehat{\psi_1}(\lambda)\widehat{\overline{\psi}_2}(\lambda) = \langle \psi_1, \phi_\lambda \rangle \langle \phi_\lambda, \psi_2 \rangle. \tag{2.5}$$

3 Expanding Horospheres in $\mathbb{H}^{1+1}$

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ and assume $\delta > 1/2$. Suppose the discrete spectrum of Γ has k many values above the base, that is, there exist eigenvalues $\delta(1-\delta) = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \frac{1}{4}$. Further, write $\lambda_k = s_k(1-s_k)$. The purpose of this section is to prove the following equidistribution theorem, which is the $n = 1$ case of Theorem 1 for K-invariant functions.

Theorem 5 (Horocyclic Equidistribution). *Suppose $\Gamma_H \backslash H$ is closed. Let $F \in C_c^\infty(\Gamma \backslash \mathbb{H})$, then the horocyclic average satisfies*

$$\mathcal{M}_T(F) = c_0 T^{\delta-1} + c_1 T^{s_1-1} + \cdots + c_k T^{s_k-1} + O(T^{\delta/3-2/3}(\log T)^{2/3}) \tag{3.1}$$

where c_i depend only on the group Γ and F, for all $i = 0, 1, \ldots, k$.

Remark. For the modular surface we recover a weaker version of Sarnak and Zagier's results [Sar81, Theorem 1] [Zag81] which do not rely on any smoothing procedure, but exploit the exact form of the modular surface.

If $\Gamma_H \backslash H$ is closed then we can assume the cusp at ∞ has width $[-X, X]$. If not, then since F has compact support, we can assume the support is contained in $x \in [-X, X]$. Then the average becomes

$$\int_{-X}^{X} F(x + i/T) \mathrm{d}x$$

where $\mathcal{F}$ is a fundamental domain for Γ.

Let

$$\psi_{T,\varepsilon}(y) := \frac{1}{2\varepsilon} \mathbb{1}(y \in [1/T - \varepsilon/T^2, 1/T + \varepsilon/T^2])$$

be the L^1 normalized indicator function. We write $\psi_{\varepsilon,T}(z) = \psi_{\varepsilon,T}(y)$ for $z = x + iy$. Further, let $\Psi_{\varepsilon,T} : \Gamma \backslash \mathbb{H} \to \mathbb{R}$ denote the automorphization of $\psi_{\varepsilon,T}$, that is

$$\Psi_{\varepsilon,T}(x + iy) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi_{\varepsilon,T}(\gamma z),$$

where Γ_∞ could be trivial. Now define the ε-thickened average to be

$$\mathcal{M}_\varepsilon(T, F) = \mathcal{M}_\varepsilon(T) := \int_{\mathcal{F}} F(x + iy) \Psi_{\varepsilon,T}(x + iy) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

3.1 Differential Equation

Now by unfolding we arrive at

$$\mathcal{M}_\varepsilon(T) = \langle F, \Psi_{\varepsilon,T} \rangle_{\Gamma \backslash \mathbb{H}}$$

$$= \int_\mathcal{F} F(z) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi_{\varepsilon,T}(\gamma z) \mathrm{d}z$$

$$= \int_{\Gamma_\infty \backslash \mathbb{H}} F(z) \psi_{\varepsilon,T}(z) \mathrm{d}z$$

$$= \int_\mathbb{R} \psi_{\varepsilon,T}(y) \int_{-X}^X F(x+iy) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Let $f(y) := \int_{-X}^X F(x+iy)\mathrm{d}x$, fix $\lambda = s(1-s) > 0$ and let $g(y) := y^2 \partial_{yy} f(y) - \lambda f(y)$. Then f satisfies the differential equation

$$y^2 \partial_{yy} f(y) - \lambda f(y) = g(y) = \int_{-X}^X (\Delta - \lambda) F \mathrm{d}x.$$

Hence f satisfies [Kon09, Lemma B.1]. That is, if $\lambda \neq 1/4$ there exist constants A, B and functions u, v such that

$$f(y) = Ay^s + By^{1-s} + u(y)y^s + v(y)y^{1-s}. \tag{3.2}$$

When $\lambda = 1/4$, we may similarly conclude that

$$f(y) = Ay^{1/2} + By^{1/2} \log y + u(y)y^{1/2} + v(y)y^{1/2} \log y. \tag{3.3}$$

For simplicity we henceforth assume that $\lambda \neq 1/4$ (if $\lambda = 1/4$ the same argument can be applied – see [Kon09]). In this case we have

$$u(y) = (1-2s)^{-1} \int_{\frac{1}{T} - \frac{\varepsilon}{T^2}}^y w^{-1-s} g(w) \mathrm{d}w, \qquad v(y) = (2s-1)^{-1} \int_{\frac{1}{T} - \frac{\varepsilon}{T^2}}^y w^{s-2} g(w) \mathrm{d}w.$$

Now we integrate $f(y)$ against $\frac{\psi_{\varepsilon,T}}{y^2}$. First, for the main term we can apply a power series expansion around $1/T$ to conclude

$$\int_0^\infty \psi_{\varepsilon,T}(y)(Ay^s + By^{1-s}) \frac{\mathrm{d}y}{y^2} = A\alpha(T) + B\beta(T),$$

where

$$\alpha(T) := \int_0^\infty \psi_{\varepsilon,T}(y) y^s \frac{\mathrm{d}y}{y^2} = \varepsilon^{-1} \left((\frac{1}{T} - \frac{\varepsilon}{T^2})^{s-1} - (\frac{1}{T} + \frac{\varepsilon}{T^2})^{s-1} \right) = T^{-s} + O(\varepsilon T^{-(s+1)}),$$

$$\beta(T) := \int_0^\infty \psi_{\varepsilon,T}(y) y^{1-s} \frac{\mathrm{d}y}{y^2} = \varepsilon^{-1} \left((\frac{1}{T} - \frac{\varepsilon}{T^2})^{-s} - (\frac{1}{T} + \frac{\varepsilon}{T^2})^{-s} \right) = T^{s-1} + O(\varepsilon T^{s-2}).$$

$$\tag{3.4}$$

As for the contribution from u, let
$$I = \int_0^\infty \psi_{\varepsilon,T}(y) y^s u(y) \frac{\mathrm{d}y}{y^2},$$

then, by integrating by parts we have (with $y_{\min} = 1/T - \varepsilon/T^2$ and $y_{\max} = 1/T + \varepsilon/T^2$)
$$I = c \int_{y_{\min}}^{y_{\max}} g(y) \frac{\mathrm{d}y}{y^2} + c y_{\max}^{s-1} \int_{y_{\min}}^{y_{\max}} y^{-1-s} g(y) \mathrm{d}y.$$

Now apply Cauchy-Schwarz, yielding
$$I \ll_{\lambda,T,\varepsilon} \int_{y_{\min}}^{y_{\max}} \int_{-X}^{X} y^{-s} \left(\frac{(\Delta - \lambda) F(z)}{y} \right) \mathrm{d}x \mathrm{d}y,$$
$$\ll_{\lambda,T,\varepsilon} \left(\int_{y_{\min}}^{y_{\max}} \int_{-X}^{X} \left| y^{-s} \right|^2 \mathrm{d}x \mathrm{d}y \right)^{1/2} \left(\int_{y_{\min}}^{y_{\max}} \int_{-X}^{X} \left| \frac{(\Delta - \lambda) F(z)}{y} \right|^2 \mathrm{d}x \mathrm{d}y \right)^{1/2},$$
$$\ll_{\lambda,T,\varepsilon} \|(\Delta - \lambda) F\|_\Gamma.$$

The same bound can be derived for the term coming from $v(y)$. Hence we conclude that for any F, we have
$$\mathcal{M}_\varepsilon(T, F) = A\alpha(T) + B\beta(T) + O(\|(\Delta - \lambda)F\|). \tag{3.5}$$

3.2 Inserting the Laplacian

Now suppose F *were* an eigenfunction of Δ with eigenvalue λ. Then the final term in (3.5) vanishes. Plugging in two values $T = 1$ and $T = b$, we can solve for A and B. A short calculation shows (for $\lambda \neq 1/4$)
$$\mathcal{M}_\varepsilon(T, F) = K_T(s) \mathcal{M}_\varepsilon(1, F) + L_T(s) \mathcal{M}_\varepsilon(b, F), \tag{3.6}$$

where
$$K_T(s) := \frac{\beta(b)\alpha(T) - \alpha(b)\beta(T)}{\alpha(1)\beta(b) - \alpha(b)\beta(1)}, \qquad L_T(s) := \frac{\alpha(1)\beta(T) - \beta(1)\alpha(T)}{\alpha(1)\beta(b) - \alpha(b)\beta(1)}. \tag{3.7}$$

Note that since $s \in (1/2, 1]$ we have $K_T(s), L_T(s) \asymp T^{s-1}$. Moreover, when $\lambda \geq 1/4$ then we have $K_T(1/2), L_T(1/2) \ll T^{-1/2} \log T$.

We abuse notation and write $K_T(\lambda) = K_T(s)$ where $\lambda = s(1 - s)$ and likewise for L_T. Furthermore, just as one can define the exponential of a matrix, we let $K_T(\Delta)$ and $L_T(\Delta)$ be the analogous operators for $K_T(s)$ and $L_T(s)$ respectively. The following theorem states that (3.6) holds even if F is not an eigenfunction of the Laplacian.

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Theorem 6 (Main Identity)**.** *For fixed $T \geq 1$, there exists a number, b, such that*

$$\Psi_{\varepsilon,T} = K_T(\Delta)\Psi_{\varepsilon,1} + L_T(\Delta)\Psi_{\varepsilon,b} \tag{3.8}$$

holds almost everywhere. Moreover K_T and L_T satisfy the bounds

$$K_T(s), L_T(s) \ll \begin{cases} T^{s-1} & \text{if } s \in (1/2, 1] \\ T^{-1/2}\log T & \text{if } s = 1/2 + it. \end{cases} \tag{3.9}$$

Proof. The proof follows an almost verbatim application of [Kon09, Proof of Theorem 3.2]. □

3.3 Proof of Thickened Equidistribution

To simplify notation, assume there are no eigenvalues other than the base eigenvalue λ_0. Note that by applying the abstract Parseval's identity

$$\begin{aligned} \mathcal{M}_\varepsilon(T) &= \langle \Psi_{\varepsilon,T}, F \rangle_\Gamma \\ &= \langle \widehat{\Psi_{\varepsilon,T}}, \widehat{F} \rangle_{\mathrm{Spec}(\Gamma)} \\ &= \widehat{F}(\lambda_0)\widehat{\Psi_{\varepsilon,T}}(\lambda_0) + \int_{\mathrm{Spec}(\Gamma)\backslash\{\lambda_0\}} \widehat{F}(\lambda)\widehat{\Psi_{\varepsilon,T}}(\lambda)\mathrm{d}\widetilde{\mu}(\lambda). \end{aligned}$$

Using Patterson-Sullivan theory we know what the base eigenfunction is, and thus, since F is compactly supported, and thus integrable with respect to the Burger-Roblin measure we conclude that

$$\widehat{F}(\lambda_0) := \langle F, \phi_0 \rangle$$

is some finite, explicit constant. Furthermore, using Theorem 6 and (3.4) we have

$$\widehat{\Psi_{\varepsilon,T}}(\lambda_0) = T^{\delta-1}(c_1\langle \Psi_{\varepsilon,1}, \phi_0 \rangle + c_2\langle \Psi_{\varepsilon,b}, \phi_0 \rangle) + O(\varepsilon T^{-\delta}).$$

Hence, since $\Psi_{\varepsilon,1}$ can be unfolded, and $\psi_{\varepsilon,1}$ has unit mass, we can use the mean value theorem (as done in [Kon09, (4.16)]) to conclude

$$\langle \Psi_{\varepsilon,1}, \phi_0 \rangle = C(1 + O(\varepsilon)).$$

From which it follows that

$$\widehat{F}(\lambda_0)\widehat{\Psi_{\varepsilon,T}}(\lambda_0) = C_{\Gamma,F}T^{\delta-1}(1 + O(\varepsilon)).$$

The lower order contribution from any exceptional eigenvalues can be similarly estimated to give $c_i T^{s_i-1}(1 + O(\varepsilon))$.

As for the error term, we may apply Theorem 6, *ii)* from the abstract spectral theorem (Theorem 4) and the estimate (3.9), yielding

$$
\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F}(\lambda)\widehat{\Psi_{\varepsilon,T}}(\lambda)\mathrm{d}\widetilde{\mu}(\lambda) = \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F}(\lambda)(K_T\widehat{(\Delta)\Psi}_{\varepsilon,1}(\lambda) + L_T\widehat{(\Delta)\Psi}_{\varepsilon,b}(\lambda)\mathrm{d}\widetilde{\mu}(\lambda)
$$

$$
= \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F}(\lambda)(K_T(\lambda)\widehat{\Psi_{\varepsilon,1}}(\lambda) + L_T(\lambda)\widehat{\Psi_{\varepsilon,b}}(\lambda)\mathrm{d}\widetilde{\mu}(\lambda)
$$

$$
\ll T^{-1/2}\log T \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F}(\lambda)(\widehat{\Psi_{\varepsilon,1}}(\lambda) + \widehat{\Psi_{\varepsilon,b}}(\lambda))\mathrm{d}\widetilde{\mu}(\lambda).
$$

To conclude we apply the abstract Parseval's identity and Cauchy-Schwarz giving

$$
\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F}(\lambda)\widehat{\Psi_{\varepsilon,T}}(\lambda)\mathrm{d}\widetilde{\mu}(\lambda) \ll T^{-1/2}\log T\|F\|_\Gamma\|\Psi_{\varepsilon,1}\|_\Gamma.
$$

This is the source of the $\|F\|_\Gamma$ in the error term. As for the second factor we have

$$
\|\Psi_{\varepsilon,1}\|_\Gamma \ll \frac{1}{\sqrt{\varepsilon}}.
$$

Leading to the following 'thickened' version of the equidistribution result

$$
\mathcal{M}_\varepsilon(T,F) = c_0 T^{\delta-1} + c_1 T^{s-1} + \ldots c_k T^{s_k-1} + O(\varepsilon^{-1/2}T^{-1/2}\log T), \tag{3.10}
$$

where c_i depend on the group, ε, and on F, moreover, for each i we can write $c_i = C_i(1 + O(\varepsilon))$, where C_i is independent of ε.

3.4   Proof of Theorem 5

By the unit mass and positivity of Ψ, we have that, after unfolding

$$
|\mathcal{M}(T) - \mathcal{M}_\varepsilon(T)| = \left| \int_{1/T-\varepsilon/T^2}^{1/T+\varepsilon/T^2} \int_{-X}^{X} (F(x+i/T) - F(z))\psi_{\varepsilon,T}(z)\frac{\mathrm{d}x\mathrm{d}y}{y^2} \right|.
$$

Now apply Cauchy-Schwarz

$$
\ll \left(\int_{1/T-\varepsilon/T^2}^{1/T+\varepsilon/T^2} \int_{-X}^{X} (F(x+i/T) - F(z))^2\mathrm{d}z \right)^{1/2} \left(\int_{1/T-\varepsilon/T^2}^{1/T+\varepsilon/T^2} \int_{-X}^{X} \psi_{\varepsilon,T}(z)^2\mathrm{d}z \right)^{1/2},
$$

$$
\ll \frac{1}{\sqrt{\varepsilon}} \left(\int_{1/T-\varepsilon/T^2}^{1/T+\varepsilon/T^2} \int_{-X}^{X} (F(x+i/T) - F(z))^2\mathrm{d}z \right)^{1/2}.
$$

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Finally apply the mean value theorem and the fact that F is assumed to be compactly supported to conclude

$$|\mathcal{M}(T) - \mathcal{M}_\varepsilon(T)| \ll \frac{\|F\|_{1,\infty}}{\sqrt{\varepsilon}} \left(\varepsilon^2 T^{2(\delta-1)} \int_{1/T-\varepsilon/T^2}^{1/T+\varepsilon/T^2} \int_{-X}^{X} 1 \mathrm{d}z \right)^{1/2} \qquad (3.11)$$

$$\ll \varepsilon T^{\delta-1} \|F\|_{1,\infty},$$

the factor of $T^{\delta-1}$ comes from the fact that F is compactly supported in $\Gamma\backslash\mathbb{H}$. Namely, since F has compact support, the only fundamental domains which intersect a small neighborhood around the horosphere of height $1/T$ and thus contribute to the above integral are those with maximum height near $1/T$. There are approximately T^δ such domains. Each such domain contributes T^{-1} to the total integral. Thus we arrive at $T^{\delta-1}$ (this is a standard loss of mass argument).

Finally to prove Theorem 5 we put (3.10) and (3.11) together, giving

$$\mathcal{M}(T, F) = c_0 \mu(F) T^{\delta-1} + c_1 T^{s-1} + \dots c_k T^{s-k} + O(\varepsilon^{-1/2} T^{-1/2} \log T \|F\|_\Gamma) \qquad (3.12)$$

$$+ O(\varepsilon T^{\delta-1} \|F\|_{1,\infty}).$$

Now choose $\varepsilon = T^{1/2-2\delta/3} (\log T)^{2/3} \|F\|_\Gamma^{2/3} \|F\|_{1,\infty}^{-2/3}$ to conclude (3.1).

4 Expanding Horospheres – Rank 1

In this section we present the proof of Theorem 1 for general n. To apply our method, we first thicken in the y-direction, then we additionally mollify in the $\overline{K}$-direction. To that end, fix a value $\varepsilon > 0$ to be determined later (depending on T). As we did when $n = 1$ let $\psi_{\varepsilon,T}(a_y)$ denote the scaled indicator function

$$\psi_{\varepsilon,T}(a_y) := \frac{1}{2\varepsilon} \mathbb{1}\left(y \in \left[\frac{1}{T} - \frac{\varepsilon}{T^{n+1}}, \frac{1}{T} - \frac{\varepsilon}{T^{n+1}} \right] \right).$$

Thus $\psi_{\varepsilon,T}$ is normalized against $\frac{\mathrm{d}y}{y^{n+1}}$. Let $\psi_{\varepsilon,T}(hak) = \psi_{\varepsilon,T}(a)$ and let

$$\Psi_{\varepsilon,T}(g) := \sum_{\gamma \in \Gamma_H \backslash \Gamma} \psi_{\varepsilon,T}(\gamma g)$$

denote the automorphization of $\psi_{\varepsilon,T}$. Furthermore, let $\psi_{\varepsilon,K}(k)$ be a unit mass, even, bump function supported in a ball of radius ε around $k = e$. Given $F \in L^2(\Gamma\backslash G/M)$ let

$$\widetilde{F} := \int_{\overline{K}} F(gk^{-1}) \psi_{\varepsilon,\overline{K}}(k) \, \mathrm{d}k.$$

Now write

$$\mathcal{M}_\varepsilon(T) := \langle \Psi_{\varepsilon,T}, \widetilde{F} \rangle_\Gamma$$

$$= \int_{\Gamma\backslash G} \Psi_{\varepsilon,T}(g)\widetilde{F}(g) \ \mathrm{d}g.$$

4.1 Unfolding to derive the differential equation

Let $\mathcal{X}$ denote the range of $\mathbf{x}$ in the support of $\widetilde{F}$. Note that $\widetilde{F}$ has compact support since F does, thus a particular coordinate of $\mathcal{X}$ either represents the periodic length of the fundamental domain in a particular direction, or some finite interval inside of a non-compact direction. Now unfold the inner product:

$$\mathcal{M}_\varepsilon(T) = \int_{\Gamma\backslash G} \widetilde{F}(g)\Psi_{\varepsilon,T}(g)\mathrm{d}g$$

$$= \int_{\Gamma\backslash G} \widetilde{F}(g) \sum_{\Gamma_H\backslash\Gamma} \psi_{\varepsilon,T}(\gamma g)\mathrm{d}g$$

$$= \int_{\Gamma_H\backslash G} \widetilde{F}(g)\psi_{\varepsilon,T}(g)\mathrm{d}g$$

$$= \int_0^\infty \psi_{\varepsilon,T}(y) \int_{\mathcal{X}} \int_K \widetilde{F}(h(\mathbf{x})a_y k) \ \frac{\mathrm{d}k\mathrm{d}\mathbf{x}\mathrm{d}y}{y^{n+1}}.$$

We treat $\widetilde{F}$ as a general $L^2(\Gamma\backslash G/M)$-function, and let

$$f(y) := \int_{\mathcal{X}} \int_K \widetilde{F}(h(\mathbf{x})a_y k) \ \mathrm{d}k\mathrm{d}\mathbf{x}.$$

then $f(y)$ satisfies

$$(y^2\partial_{yy} - (n-1)y\partial_y)f(y) = \int_{\mathcal{X}} \int_K (\mathcal{C} - \lambda)\widetilde{F}(h(\mathbf{x})a_y k) \ \mathrm{d}k\mathrm{d}\mathbf{x} =: g(y). \qquad (4.1)$$

This follows since applying the Casimir operator is done through right multiplication, thus, integrating with respect to k reduces to the K-invariant case: i.e

$$\int_K \mathcal{C}\widetilde{F}(ha_y k)\mathrm{d}k = \Delta \int_K \widetilde{F}(ha_y k)\mathrm{d}k.$$

then we notice that we can use integration by parts to eliminate all the derivatives in x_i coming from the Laplacian and since $\widetilde{F}$ is a compactly supported, smooth, function on $\Gamma\backslash G/M$ it follows that all boundary terms are 0. This leaves (4.1).

From here we apply a standard argument from the calculus of variations to conclude the following

Theorem 7. *For any $s > 0$ with $s \in (n/2, n)$, there exist constants A, B such that*

$$f(y) = Ay^s + By^{n-s} + y^s u(y) + y^{n-s} v(y) \tag{4.2}$$

where

$$u(y) := (n - 2s)^{-1} \int_y^{1/T + \varepsilon/T^{n+1}} w^{n-s+2} g(w) \frac{\mathrm{d}w}{w^{n+1}}, \qquad and$$

$$v(y) := -(n - 2s)^{-1} \int_y^{1/T + \varepsilon/T^{n+1}} w^{s+2} g(y) \frac{\mathrm{d}w}{w^{n+1}}$$

Moreover, when $s = n/2 + it$ then

$$f(y) = Ay^{n/2} + By^{n/2} \log y + y^{n/2} u(y) + y^{n/2} v(y) \log y \tag{4.3}$$

where

$$u(y) := \int_y^{1/T + \varepsilon/T^{n+1}} w^{n/2+2} \log(w) g(w) \frac{\mathrm{d}w}{w^{n+1}} \qquad and$$

$$v(y) := - \int_y^{1/T + \varepsilon/T^{n+1}} w^{n/2+2} g(w) \frac{\mathrm{d}w}{w^{n+1}}.$$

Proof. The proof of this theorem is classical and the details can be found in [Kon09, Appendix B]. We omit the proof here. □

From here we can apply the integral in y and use the same Cauchy-Schwarz argument, as used above to derive (3.5), to write

$$\mathcal{M}_\varepsilon(T, F) = A\alpha(T) + B\beta(T) + O_{T,\varepsilon,\Gamma}(\|(\mathcal{C} - \lambda)\widetilde{F}\|_{L^2(\Gamma \backslash G)}) \tag{4.4}$$

where, if $s \in (n/2, n)$

$$\alpha(T) := \frac{1}{2\varepsilon} \int_{1/T - \varepsilon/T^{n+1}}^{1/T + \varepsilon/T^{n+1}} y^{s - (n+1)} \mathrm{d}y = T^{-s} + O(\varepsilon T^{-s-1})$$

$$\beta(T) := \frac{1}{2\varepsilon} \int_{1/T - \varepsilon/T^{n+1}}^{1/T + \varepsilon/T^{n+1}} y^{n-s-(n+1)} \mathrm{d}y = T^{s-n} + O(\varepsilon T^{s-n-1}),$$

and if $s = n/2 + it$

$$\alpha(T) := \frac{1}{2\varepsilon} \int_{1/T - \varepsilon/T^{n+1}}^{1/T + \varepsilon/T^{n+1}} y^{n/2 - (n+1)} \mathrm{d}y = T^{-n/2} + O(\varepsilon T^{-n/2-1})$$

$$\beta(T) := \frac{1}{2\varepsilon} \int_{1/T - \varepsilon/T^{n+1}}^{1/T + \varepsilon/T^{n+1}} y^{n/2 - s - (n+1)} \log(y) \mathrm{d}y = T^{-n/2} \log(T) + O(\varepsilon T^{-n/2-1} \log(T)).$$

4.2   Inserting the Laplacian

Once again, fix a real number $b > 0$, let

$$K_T(s) := \frac{\beta(b)\alpha(T) - \alpha(b)\beta(T)}{\alpha(1)\beta(b) - \alpha(b)\beta(1)}, \qquad L_T(s) := \frac{\alpha(1)\beta(T) - \beta(1)\alpha(T)}{\alpha(1)\beta(b) - \alpha(b)\beta(1)},$$

and note that $K_T(s), L_T(s) \asymp T^{s-n}$ when $s \neq n/2$ and $K_T(s), L_T(s) \ll T^{-n/2}\log(T)$ when $s = n/2 + it$. The following theorem (analogous to Theorem 6) shows that we can 'grow' $\Psi_{\varepsilon,1}$ to height T using the Casimir operator

Theorem 8. *For fixed $T \geq 1$ there exists a number $b > 0$ and corresponding K_T and L_T such that*

$$\Psi_{\varepsilon,T} = K_T(\mathcal{C})\Psi_{\varepsilon,1} + L_T(\mathcal{C})\Psi_{\varepsilon,b} \tag{4.5}$$

almost everywhere. Moreover K_T and L_T satisfy the following bounds

$$K_T(s), L_T(s) \ll \begin{cases} T^{s-n} & \text{if } s \in (n/2, n) \\ T^{-n/2}\log T & \text{if } s = n/2 + it. \end{cases} \tag{4.6}$$

Proof. First define the difference function

$$G_T := \Psi_{\varepsilon,T} - K_T(\mathcal{C})\Psi_{\varepsilon,1} + L_T(\mathcal{C})\Psi_{\varepsilon,b}. \tag{4.7}$$

We can use exactly the argument in [Kon09, Proof of Proposition 3.5] to show that for any $\widetilde{F} \in L^2(\Gamma\backslash G/M)$ we have that

$$\langle G_T, \widetilde{F}\rangle \ll_{\lambda,T} \|(\mathcal{C} - \lambda)\widetilde{F}\|_\Gamma. \tag{4.8}$$

This follows by the construction of K_T and L_T and (4.4), we omit the details.

Now, any function G_T satisfying (4.8) for every λ and any $\widetilde{F} \in L^2(\Gamma\backslash G/M)$ must vanish almost everywhere by the same argument at [Kon09, Proof of Theorem 3.2]. $\qquad\square$

4.3   Proof of Theorem 1

Now, to prove Theorem 1 we first exploit the abstract spectral theorem to pass to the spectrum. That is by the abstract Parseval's identity (2.2)

$$\begin{aligned} \mathcal{M}_\varepsilon(T, F) &= \langle \widetilde{F}, \Psi_{\varepsilon,T}\rangle_\Gamma \\ &= \langle \widehat{\widetilde{F}}, \widehat{\Psi_{\varepsilon,T}}\rangle_{\text{Spec}} \\ &= \widehat{\widetilde{F}}(\lambda_0)\widehat{\Psi_{\varepsilon,T}}(\lambda_0) + \int_{\text{Spec}\backslash\{\lambda_0\}} \widehat{\widetilde{F}}(\lambda)\widehat{\Psi_{\varepsilon,T}}(\lambda)\mathrm{d}\widetilde{\mu}(\lambda). \end{aligned} \tag{4.9}$$

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First, let us analyze the term coming from the base eigenvalue. Using (2.4) and our main identity (4.5) we have that

$$\widehat{\widetilde{F}}(\lambda_0)\widehat{\widetilde{\Psi}_{\varepsilon,T}}(\lambda_0) = K_T(\lambda_0)\langle \operatorname{Proj}_{\mathscr{H}_0}(\widetilde{F}), \operatorname{Proj}_{\mathscr{H}_0}(\Psi_{\varepsilon,1})\rangle + L_T(\lambda_0)\langle \operatorname{Proj}_{\mathscr{H}_0}(\widetilde{F}), \operatorname{Proj}_{\mathscr{H}_0}(\Psi_{\varepsilon,b})\rangle$$
$$= T^{\delta-n}c\langle \operatorname{Proj}_{\mathscr{H}_0}(\widetilde{F}), \operatorname{Proj}_{\mathscr{H}_0}(\Psi_{\varepsilon,1}) + \operatorname{Proj}_{\mathscr{H}_0}(\Psi_{\varepsilon,b})\rangle + O(T^{-\delta}).$$

The projection $\operatorname{Proj}_{\mathscr{H}_0}(\Psi_{\varepsilon,1})$ can be expressed using a Burger-Roblin type measure, as done in [MO15, p. 861]. From whence it follows that $\operatorname{Proj}_{\mathscr{H}_0}(\Psi_{\varepsilon,1}) = C + O(\varepsilon)$. Similarly, we can use our understanding of the base eigenspace to conclude that $\operatorname{Proj}_{\mathscr{H}_0}(\widetilde{F}) = C(1 + O(\varepsilon))$ for some C depending only on F (this comes from the smoothing in $\overline{K}$ and another mean value theorem argument). From here, we conclude that

$$\widehat{\widetilde{F}}(\lambda_0)\widehat{\widetilde{\Psi}_{\varepsilon,T}}(\lambda_0) = cT^{\delta-n}(1 + O(\varepsilon)) + O(T^{-\delta}).$$

Note that we are unable to extract the ε-dependence of the terms coming from larger eigenvalues. In thickened form (i.e ε fixed) we can apply the same process and extract lower order terms. However at present we have no way to extract the ε-dependence of $\operatorname{Proj}_{\mathscr{H}_i}(\Psi_{\varepsilon,1})$, since $\mathscr{H}_i$ has infinite multiplicity. Note that this is not the case if F is right K-invariant, this is why, in that case, we can extract all lower order terms coming from the exceptional spectrum.

As for the remainder of the spectrum we can apply Parseval and Cauchy-Schwarz in the same way as we did previously, arriving at

$$\int_{\operatorname{Spec}\setminus\{\lambda_0\}} \widehat{\widetilde{F}}(\lambda)\widehat{\widetilde{\Psi}_{\varepsilon,T}}(\lambda)\, d\widetilde{\mu}(\lambda) \ll \max\{T^{s_1-n}, T^{-n/2}\log T\}\|F\|_\Gamma\|\Psi_{\varepsilon,1}\|_\Gamma$$

Now we note that $\|\Psi_{\varepsilon,1}\| = O(\varepsilon^{-(\dim(\overline{K})+1)/2}) = O\left(\varepsilon^{-\frac{n^2-3n+6}{4}}\right)$.

From here deduce (let us assume for simplicity $s_1 < n/2$

$$\mathcal{M}_\varepsilon(T,F) = cT^{\delta-n}(1 + O(\varepsilon)) + O\left(\varepsilon^{-\frac{n^2-3n+6}{4}}\|F\|_\Gamma T^{-n/2}\log T\right).$$

By the same mean value principle argument as previously used

$$|\mathcal{M}_\varepsilon(T,F) - \mathcal{M}_T(F)| \ll T^{\delta-n}\varepsilon\|F\|_{1,\infty}. \tag{4.10}$$

Choosing $\varepsilon = T^{-\frac{4(\delta-n/2)}{P(n)}}(\|F\|_\Gamma\|F\|_{1,\infty}^{-1}\log T)^{\frac{4}{P(n)}}$,

$$\mathcal{M}_\varepsilon(T,F) = cT^{\delta-n} + O\left((T^{-\frac{4(\delta-n/2)}{P(n)}}(\log T)^{\frac{4}{P(n)}})T^{\delta-n}\|F\|_\Gamma^{\frac{4}{P(n)}}\|F\|_{1,\infty}^{\frac{P(n)-4}{P(n)}}\right).$$

5   Preliminaries in higher rank

Now let $G := \mathrm{SL}_n(\mathbb{R})$ for $n > 2$.

Decomposition of $L^2(\Gamma\backslash G/K)$ into irreducibles and the spectral theorem: Let $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ have finite co-volume, let $\mathscr{H} := L^2(\Gamma\backslash G/K)$ be the L^2 Hilbert space. The Casimir operators, $\Delta_1, \ldots, \Delta_{n-1}$ are positive, self-adjoint operators acting on the L^2 Hilbert space. Thus the spectrum of each lies in $\mathbb{R}_{>0}$. We could apply the abstract spectral theorem to each individually, however, for our purposes this is not enough. We will need to consider the joint spectrum and spectral measure coming from the unitary dual representation.

The group G acts by right regular representation on $\mathscr{H}$. The Hilbert space $\mathscr{H}$ decomposes into components as follows

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k \oplus \mathscr{H}^{tempered}$$

where $\mathscr{H}_i$ is a finite dimensional eigenspace with Δ_j-eigenvalue $\lambda_j^{(i)}$ and $\mathscr{H}^{tempered}$ denotes the tempered spectrum.

Now there exists a measure $\widetilde{\mu}$ on the space of irreducible representations such that

$$f = \int \widehat{f}(\boldsymbol{\lambda}) \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda})$$

(see [CSM95, p. 122]), where $\widehat{f}$ denotes the projection of f onto the irreducible parameterized by $\boldsymbol{\lambda}$. Moreover, Harish-Chandra showed that this measure satisfies Plancherel's identity (see [CSM95, p. 122])

$$\int_{\Gamma\backslash G} |f|^2 \, \mathrm{d}g = \int_0^\infty |\widehat{f}|^2 \mathrm{d}\widetilde{\mu}, \tag{5.1}$$

and by the polarization formula: $\langle f, g \rangle = \frac{1}{2}(\|f\|^2 + \|g\|^2 - \|f - g\|^2)$, Plancherel's identity leads to Parseval's identity:

$$\langle f, g \rangle_\Gamma = \int_0^\infty \widehat{f}\,\widehat{g} \, \mathrm{d}\widetilde{\mu}. \tag{5.2}$$

Finally, by Schur's lemma [CSM95, p. 84] the Casimir operators each act via multiplication on the irreducible representations: that is, given an irreducible representation we can associate a value $\boldsymbol{\lambda}$, then Δ_i acts via a scalar

$$\widehat{\Delta_i f}(\boldsymbol{\lambda}) = \lambda_i \widehat{f}(\boldsymbol{\lambda}). \tag{5.3}$$

Let $\mathcal{S}$ denote the collection of scalars of $\Delta_1, \ldots, \Delta_{n-1}$ acting on the different irreducible subspaces (without multiplicity). Thus

$$\mathcal{S} = \{\boldsymbol{\lambda}^{(0)}, \boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}\} \cup \mathcal{S}^{cont},$$

where $\mathcal{S}^{cont}$ denotes the tempered spectrum.

6 Expanding Horospheres in $\mathrm{SL}_3(\mathbb{R})$

In higher rank, the proof follows the same approach with some modification; for simplicity we start with the case $n = 3$. Once again we need to smooth in the non-horospherical directions, in this case y_1 and y_2. To that end, fix a value ε (one could choose to have two smoothing parameters, but when optimizing it turns out that the optimal choice is to set them equal to each other). Let $\psi_{T,\varepsilon}$ be the indicator of the region $[1/T - \varepsilon/T^3, 1/T + \varepsilon/T^3]$ multiplied by ε^{-1}. For $\mathcal{H} \ni z = h(\mathbf{x})a(\mathbf{y})$ we write

$$\psi_{\mathbf{T},\varepsilon}(z) = \psi_{T_1,\varepsilon}(y_1)\psi_{T_2,\varepsilon}(y_2),$$

now automorphize $\psi_{\mathbf{T},\varepsilon}$ as follows

$$\Psi_{\mathbf{T},\varepsilon}(z) = \sum_{\Gamma_H \backslash \Gamma} \psi_{\mathbf{T},\varepsilon}(\gamma z).$$

Let $\mathcal{F}$ denote a fundamental domain for the group Γ. Now we define the ε-thickened horospherical average as

$$\mathcal{M}_\varepsilon(\mathbf{T}, F) := \int_{\mathcal{F}} F(z)\Psi_{\mathbf{T},\varepsilon}(z)\mathrm{d}z.$$

6.1 Differential Equation

Once again, our first step is to unfold $\mathcal{M}_\varepsilon$:

$$\mathcal{M}_\varepsilon(\mathbf{T}, F) = \int_{\Gamma_H \backslash \mathcal{H}} F(z)\psi_{\varepsilon,\mathbf{T}}(z)\mathrm{d}z$$

$$= \int_{\mathbb{R}^2_{>0}} \psi_{\varepsilon,T_1}(y_1)\psi_{\varepsilon,T_2}(y_2) \int_{\Gamma_H \backslash H} F(ha(\mathbf{y}))\mathrm{d}h\frac{\mathrm{d}y_1\mathrm{d}y_2}{y_1^3 y_2^3}.$$

Let $f(\mathbf{y}) := \int_{\Gamma_H \backslash H} F(ha(\mathbf{y}))\mathrm{d}h$ denote the inner integral.

Fix any vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Then by integration by parts, the function f satisfies the equation

$$(y_1^2\partial_{y_1 y_1} + y_2^2\partial_{y_2 y_2} - y_1 y_2\partial_{y_1 y_2} - \lambda_1)f(\mathbf{y}) = \int_{\Gamma_H \backslash H} (\Delta_1 - \lambda_1)F(ha(\mathbf{y}))\mathrm{d}h =: g_1(y_1, y_2).$$

$$(6.1)$$

Likewise, f satisfies

$$(-y_1^2 y_2 \partial_{y_1 y_1 y_2} + y_1 y_2^2 \partial_{y_1 y_2 y_2} + y_1^2 \partial_{y_1 y_1} - y_2^2 \partial_{y_2 y_2} - \lambda_2) f(\mathbf{y}) = \int_{\Gamma_H \backslash H} (\Delta_2 - \lambda_2) F(ha(\mathbf{y})) \mathrm{d}h.$$

(6.2)

In both cases $y_1^s y_2^r$ is a solution to the homogeneous case, but the first equation requires $sr - s(s-1) - r(r-1) = \lambda_1$ and the second requires $sr(s-r) + s(s-1) + r(r-1) = \lambda_2$. For a given pair λ_1, λ_2 there are at most six pairs (s, r) satisfying both equations. Let $\phi_{homo}(\mathbf{y}) := \sum_{i=1}^{6} A_i y_1^{s_i} y_2^{r_i}$ denote the homogeneous solution to *both* equations.

Let $\phi_{homo,1}$ denote a general solution to the homogeneous (6.1). Now we can use the method of Green's functions to lift this homogeneous solution to a solution of the inhomogeneous equation, that is our particular solution will take the form

$$u_p(y_1, y_2) := \int_{\xi_1, \xi_2} \mathcal{G}(y_1, y_2, \xi_1, \xi_2) g(\xi_1, \xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

Then $\mathcal{G}$ satisfies the equation

$$(y_1 y_2 \partial_{y_1 y_2} - y_1^2 \partial_{y_1 y_1} - y_2^2 \partial_{y_2 y_2} - \lambda) \mathcal{G}(y_1, y_2, \xi_1, \xi_2) = \delta(y_1 - \xi_1) \delta(y_2 - \xi_2).$$

Now assume $\mathcal{G}$ is point pair invariant, i.e it depends only on the scalar $r = |\boldsymbol{\xi} - \mathbf{y}|$. In this case, after integrating out the angular direction the above equation becomes for $r > 0$

$$\frac{\pi}{2} r^2 \partial_{rr} h(r) = 2\pi \lambda_1 h(r).$$

Hence, writing $\lambda_1 = \kappa(\kappa - 1)$ yields

$$\mathcal{G}(\mathbf{y}, \boldsymbol{\xi}) = |\boldsymbol{\xi} - \mathbf{y}|^{\kappa}.$$

Thus our full solution is

$$f(\mathbf{y}) = \phi_{homo,1}(\mathbf{y}) + u_{p,1}(y_1, y_2).$$

(6.3)

Similarly, if we apply the same argument with Δ_2 in place of Δ_1 we arrive at the equation

$$f(\mathbf{y}) = \phi_{homo,2}(\mathbf{y}) + u_{p,2}(y_1, y_2),$$

(6.4)

where $\phi_{homo,2}$ satisfies $\Delta_2 \phi_{homo,2} = 0$ and $u_{p,2}(y_1, y_2)$ is the particular solution associated to λ_2.

Thus, we can write

$$\begin{aligned} f(\mathbf{y}) &= \phi_{homo}(\mathbf{y}) + \widetilde{\phi}_{homo,1}(\mathbf{y}) + u_{p,1}(y_1, y_2) \\ &= \phi_{homo}(\mathbf{y}) + \widetilde{\phi}_{homo,2}(\mathbf{y}) + u_{p,2}(y_1, y_2), \end{aligned}$$

where $\widetilde{\phi}_{homo,i}(\mathbf{y}) = \phi_{homo,i}(\mathbf{y}) - \phi_{homo}(\mathbf{y})$, for $i = 1, 2$. Now, to integrate with respect to $\mathbf{y}$, let

$$\alpha(\mathbf{T}) := \int_0^\infty \int_0^\infty \psi_{\varepsilon,\mathbf{T}}(\mathbf{y})\phi_{homo}(\mathbf{y})y_1^{-3}y_2^{-3}\mathrm{d}y_1\mathrm{d}y_2.$$

Then we have that, by the usual Cauchy-Schwartz argument and the definitions of ϕ_{homo}, $\phi_{homo,1}$, and $\phi_{homo,2}$, for every pair λ_1 and λ_2 in $\mathbb{R}^2$ we have

$$\mathcal{M}_\varepsilon(\mathbf{T}, F) = \alpha(\mathbf{T}) + O(\|(\Delta_1 - \lambda_1)F\|) + O(\|(\Delta_2 - \lambda_2)F\|). \tag{6.5}$$

Further, for $i = 1, \ldots, 6$, let

$$\alpha_i(\mathbf{T}) := \int_0^\infty \int_0^\infty \psi_{\varepsilon,\mathbf{T}}(\mathbf{y})y_1^{s_i}y_2^{r_i}y_1^{-3}y_2^{-3}\mathrm{d}y_1\mathrm{d}y_2$$
$$= T_1^{-s_i}T_2^{-r_i} + O(\varepsilon T_1^{-s_i-1}T_2^{-r_i} + \varepsilon T_2^{-r_i-1}T_1^{-s_i}). \tag{6.6}$$

Then

$$\mathcal{M}_\varepsilon(\mathbf{T}, F) = \sum_{i=1}^6 A_i\alpha_i(\mathbf{T}) + O(\|(\Delta_1 - \lambda_2)F\|) + O(\|(\Delta_2 - \lambda_2)F\|). \tag{6.7}$$

Now we fix 6 times $\mathbf{b}_i \in \mathbb{R}_{>1}^2$ and write the matrix equation

$$(\mathcal{M}_\varepsilon(\mathbf{b}_1, F), \mathcal{M}_\varepsilon(\mathbf{b}_2, F), \ldots \mathcal{M}_\varepsilon(\mathbf{b}_6, F))^T = \mathbf{M}\mathbf{A}.$$

where $(\mathbf{M})_{ij} = \alpha_i(\mathbf{b}_j)$ and $\mathbf{A} = (A_1, A_2, \ldots, A_6)^T$. By an appropriate choice of $\mathbf{b}_i$ we can ensure that M is invertible. From here we can solve for A_i and write

$$\mathcal{M}_\varepsilon(\mathbf{T}, F) = \sum_{i=1}^6 K_{i,\mathbf{T}}(\boldsymbol{\lambda})\mathcal{M}_\varepsilon(\mathbf{b}_i, F) + O(\|(\Delta_1 - \lambda_1)F\|) + O(\|(\Delta_2 - \lambda_2)F\|) \tag{6.8}$$

for some $K_{i,\mathbf{T}}(\boldsymbol{\lambda})$ which can be made explicit. Now we are ready to prove following main identity which underpins the spectral estimates that are crucial in our later analysis. It states that in (6.8) we can replace the functions of $\boldsymbol{\lambda}$ with the same functions of $\boldsymbol{\Delta} = (\Delta_1, \Delta_2)$ (defined via power series) without affecting the error term. Then since the modified (6.8) holds for all points $\boldsymbol{\lambda}$ in the spectrum, we can in fact, show that the error vanishes.

Theorem 9 (Main Identity). *There exist constants $\mathbf{b}_i$ such that for $\mathbf{T}$ large enough*

$$\Psi_{\varepsilon,\mathbf{T}} = \sum_{i=1}^6 K_{i,\mathbf{T}}(\boldsymbol{\Delta})\Psi_{\varepsilon,\mathbf{b}_i}, \tag{6.9}$$

almost everywhere. Moreover $K_{i,\mathbf{T}}$ all satisfy,

$$K_{i,\mathbf{T}}(\boldsymbol{\lambda}) = \begin{cases} T_1^{-s_i} T_2^{-r_i} + O(\varepsilon T_1^{-s_i-1} T_2^{-r_i} + \varepsilon T_1^{-s_i} T_2^{-r_i-1}) & \text{if } s_i, r_i < 1 \\ T_1^{-1} T_2^{-1} \log(T_1) \log(T_2) & \text{if } s_i, r_i \in \{1 + it\}^2. \end{cases} \quad (6.10)$$

Proof. First, note that for any i we can apply the spectral transform and the mean value theorem to

$$\langle K_{i,\mathbf{T}}(\boldsymbol{\Delta})\Psi_{\varepsilon,\mathbf{b}}, F \rangle - K_{i,\mathbf{T}}(\boldsymbol{\lambda})\langle \Psi_{\varepsilon,\mathbf{b}}, F \rangle = \int_{\mathcal{S}} (K_{i,\mathbf{T}}(\boldsymbol{\lambda}') - K_{i,\mathbf{T}}(\boldsymbol{\lambda}))\widehat{\Psi_{\varepsilon,\mathbf{b}}}(\boldsymbol{\lambda}')\widehat{F}(\boldsymbol{\lambda}') \, \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}')$$

$$\ll \int_{\mathcal{S}} |\boldsymbol{\lambda}' - \boldsymbol{\lambda}| \, \widehat{\Psi_{\varepsilon,\mathbf{b}}}(\boldsymbol{\lambda}')\widehat{F}(\boldsymbol{\lambda}')\mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}')$$

$$\ll \int_{\mathcal{S}} (|\lambda_1' - \lambda_1| + |\lambda_2' - \lambda_2|)\widehat{\Psi_{\varepsilon,\mathbf{b}}}(\boldsymbol{\lambda}')\widehat{F}(\boldsymbol{\lambda}')\mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}')$$

$$\ll \|(\lambda_1 - \Delta_1)F\|\|\Psi_{\varepsilon,\mathbf{b}}\| + \|(\lambda_2 - \Delta_2)F\|\|\Psi_{\varepsilon,\mathbf{b}}\|.$$

From here we can use the definition of $K_{i,\mathbf{T}}$ to conclude (see [Kon09, Proof of Lemma 3.5] for details)

$$\mathcal{M}_{\varepsilon}(\mathbf{T}) = \sum_{i=1} K_{i,\mathbf{T}}(\boldsymbol{\Delta})\mathcal{M}_{\varepsilon}(\mathbf{b}_i, F) + O(\|(\Delta_1 - \lambda_1)F\|) + O(\|(\Delta_2 - \lambda_2)F\|).$$

Define for $z \in \mathcal{H}$

$$G_{\varepsilon,\mathbf{T}}(z) := \Psi_{\varepsilon,\mathbf{T}}(z) - \sum_{i=1} K_{i,\mathbf{T}}(\boldsymbol{\Delta})\Psi_{\varepsilon,\mathbf{b}_i}(z).$$

Then (6.9) will follow if we can show $G_{\varepsilon,\mathbf{T}} = 0$ almost everywhere.

Now for any function F and any point in the spectrum $\boldsymbol{\lambda}$ we have

$$\langle G_{\varepsilon,\mathbf{T}}, F \rangle = O(\|(\Delta_1 - \lambda_1)F\|) + O(\|(\Delta_2 - \lambda_2)F\|).$$

Fix a $\sigma > 0$ and a point $\boldsymbol{\lambda}$ and define F according to its spectrum:

$$\widehat{F}(\boldsymbol{\lambda}') := \begin{cases} \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') & \text{if } \boldsymbol{\lambda}' \in B_\sigma(\boldsymbol{\lambda}) \\ 0 & \text{otherwise}, \end{cases}$$

where $\widehat{G_{\varepsilon,\mathbf{T}}}$ denotes the spectral transform of $G_{\varepsilon,\mathbf{T}}$, and $B_\sigma(\boldsymbol{\lambda})$ denotes a ball of radius σ around $\boldsymbol{\lambda}$. Now apply Parseval's identity (5.2)

$$\langle G_{\varepsilon,\mathbf{T}}, F \rangle = \langle \widehat{G_{\varepsilon,\mathbf{T}}}, \widehat{F} \rangle$$

$$= \int_{B_\sigma(\boldsymbol{\lambda})} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}').$$

Using (5.3) yields

$$\|(\Delta_1 - \lambda_1)F\| \ll \left(\int_0^\infty \int_0^\infty \left| (\lambda_1' - \lambda_1)\widehat{F}(\boldsymbol{\lambda}) \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}) \right)^{1/2}$$

$$\ll \sigma \left(\int_{B_\sigma(\boldsymbol{\lambda})} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}') \right)^{1/2},$$

and similarly

$$\|(\Delta_2 - \lambda_2)F\| \ll \sigma \left(\int_{B_\sigma(\boldsymbol{\lambda})} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}') \right)^{1/2}.$$

Thus, for any σ we have

$$\int_{B_\sigma(\boldsymbol{\lambda})} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}') \ll \sigma \left(\int_{B_\sigma(\boldsymbol{\lambda})} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}') \right)^{1/2}.$$

If the left hand side vanishes we are done. If not, then

$$\int_{B_\sigma(\boldsymbol{\lambda})} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}') \ll \sigma^2 \tag{6.11}$$

for any $\boldsymbol{\lambda}$ and σ. Now let

$$f(\boldsymbol{\lambda}) := \int_{\lambda_1' < \lambda_1} \int_{|\lambda_2' - \lambda_2| < \sigma} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}')$$

then we can take a derivative

$$\frac{df}{d\lambda_1}(\lambda_1, \lambda_2) = \frac{f(\lambda_1 + \sigma, \lambda_2) - f(\lambda_1 - \sigma, \lambda_2)}{2\sigma} = \lim_{\sigma \to 0} \frac{1}{2\sigma} \int_X \left| \widehat{G}_{\varepsilon,\mathbf{T}}(\lambda_1') \right|^2 d\widetilde{\mu}(\lambda_1') = 0.$$

Thus $f'(\boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda}$, and since $f(0) = 0$ we conclude that $f = 0$ for all values of $\boldsymbol{\lambda}$.

Now let

$$\widetilde{f}(\boldsymbol{\lambda}) = \int_{\lambda_1' < \lambda_1} \int_{\lambda_2' < \lambda_2} \left| \widehat{G_{\varepsilon,\mathbf{T}}}(\boldsymbol{\lambda}') \right|^2 d\widetilde{\mu}(\boldsymbol{\lambda}')$$

then it is easy to show that $\frac{d}{d\lambda_2} \widetilde{f}(\boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda}$ and $\widetilde{f}(0) = 0$. Similarly we can deduce that $\frac{d}{d\lambda_1} \widetilde{f}(\boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda}$. Thus $\widetilde{f}(\boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda}$. From which it follows that $\widehat{G}_{\varepsilon,\mathbf{T}}$ is 0 almost everywhere.

$\square$

6.2 Proof of Theorem 2

Now, with the main identity at hand, we can proceed with the proof of Theorem 2. By Parseval's identity (5.2)

$$
\begin{aligned}
\mathcal{M}_\varepsilon(\mathbf{T}, F) &= \langle \Psi_{\varepsilon, \mathbf{T}}, F \rangle_\Gamma \\
&= \langle \widehat{\Psi_{\varepsilon, \mathbf{T}}}, \widehat{F} \rangle_{\mathrm{Spec}(\Gamma)} \\
&= \widehat{F}(\boldsymbol{\lambda}^{(0)}) \widehat{\Psi_{\varepsilon, \mathbf{T}}}(\boldsymbol{\lambda}^{(0)}) + \widehat{F}(\boldsymbol{\lambda}^{(1)}) \widehat{\Psi_{\varepsilon, \mathbf{T}}}(\boldsymbol{\lambda}^{(1)}) + \cdots + \widehat{F}(\boldsymbol{\lambda}^{(k)}) \widehat{\Psi_{\varepsilon, \mathbf{T}}}(\boldsymbol{\lambda}^{(k)}) \\
&\qquad\qquad\qquad + \int_{\mathcal{S}^{cont}} \widehat{F}(\boldsymbol{\lambda}) \widehat{\Psi_{\varepsilon, \mathbf{T}}}(\boldsymbol{\lambda}) \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}).
\end{aligned}
$$

Since F has compact support and the ith eigenspace has finite dimension, we know that the projection onto the ith eigenspace,

$$
\widehat{F}(\boldsymbol{\lambda}^{(i)}) = \sum_{k=1}^{\kappa} \langle F, \phi_{i,k} \rangle
$$

is a finite constant, where κ is the multiplicity of the ith eigenspace. Furthermore, using our bounds for $K_{i,\mathbf{T}}$, (6.10) we have

$$
\widehat{\Psi_{\varepsilon, \mathbf{T}}}(\boldsymbol{\lambda}^{(i)}) = \sum_{k=1}^{\kappa} \left(\sum_{j=1}^{6} c_j T_1^{-s_j^{(i)}} T_2^{-r_j^{(i)}} \langle \Psi_{\varepsilon, \mathbf{1}}, \phi_{i,k} \rangle \right) (1 + O(\varepsilon T_1^{-1} + \varepsilon T_2^{-1})).
$$

Moreover, by the same mean value argument we have that $\langle \Psi_{\varepsilon, \mathbf{1}}, \phi_{i,k} \rangle = C + O(\varepsilon)$. From whence it follows that

$$
\widehat{F}(\boldsymbol{\lambda}^{(i)}) \widehat{\Psi_{\varepsilon, T}}(\boldsymbol{\lambda}^{(i)}) = C_{\Gamma, F} \left(\sum_{j=1}^{6} c_j T_1^{-s_j^{(i)}} T_2^{-r_j^{(i)}} \right) (1 + O(\varepsilon)).
$$

Turning to the error term, we apply the fact that the Casimir operators are diagonal on the dual space

$$
\begin{aligned}
\int_{\mathcal{S}^{cont}} \widehat{F}(\boldsymbol{\lambda}) \widehat{\Psi_{\varepsilon, \mathbf{T}}}(\boldsymbol{\lambda}) \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}) & \\
&= \int_{\mathcal{S}^{cont}} \widehat{F}(\boldsymbol{\lambda}) \left(\sum_{i=1}^{6} K_{i,\mathbf{T}}\widehat{(\boldsymbol{\Delta})} \Psi_{\varepsilon, \mathbf{b}_i}(\boldsymbol{\lambda}) \right) \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}) \\
&= \int_{\mathcal{S}^{cont}} \widehat{F}(\boldsymbol{\lambda}) \left(\sum_{i=1}^{6} K_{i,\mathbf{T}}(\boldsymbol{\lambda}) \widehat{\Psi_{\varepsilon, \mathbf{b}_i}}(\boldsymbol{\lambda}) \right) \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}) \\
&\ll T_1^{-1} T_2^{-1} \log T_1 \log T_2 \int_{\mathcal{S}^{cont}} \widehat{F}(\boldsymbol{\lambda}) \left(\sum_{i=1}^{6} \widehat{\Psi_{\varepsilon, \mathbf{b}_i}}(\boldsymbol{\lambda}) \right) \mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}).
\end{aligned}
$$

To conclude we apply the abstract Parseval's identity and Cauchy-Schwarz giving

$$\int_{\mathcal{S}^{cont}} \widehat{F}(\boldsymbol{\lambda})\widehat{\overline{\Psi_{\varepsilon,\mathbf{T}}}}(\boldsymbol{\lambda})\mathrm{d}\widetilde{\mu}(\boldsymbol{\lambda}) \ll T_1^{-1}T_2^{-1}\log T_1 \log T_2 \|F\|_\Gamma \|\Psi_{\varepsilon,\mathbf{1}}\|_\Gamma.$$

Since F is assumed to be in $L^2(\Gamma\backslash\mathcal{H})$ its norm is bounded. As for the second term we have

$$\|\Psi_{\varepsilon,\mathbf{1}}\|_\Gamma \ll \frac{1}{\varepsilon}.$$

Leading to the following 'thickened' version of the equidistribution result

$$\mathcal{M}_\varepsilon(\mathbf{T},F) = c_0 m_\mathbf{T}(\boldsymbol{\lambda}^{(0)}) + c_1 m_\mathbf{T}(\boldsymbol{\lambda}^{(1)}) + \cdots + c_k m_\mathbf{T}(\boldsymbol{\lambda}^{(k)})$$
$$+ O(\varepsilon^{-1}T_1^{-1}T_2^{-1}\log T_1 \log T_2 \|F\|_\Gamma), \quad (6.12)$$

where, given a $\boldsymbol{\lambda}$ in the spectrum $m_\mathbf{T}(\lambda_1) := \sum_{i=1}^6 T_1^{s_i}T_2^{r_i}$. Note that the c_i depend on the group, ε and on F. Moreover for $i = 0,\ldots,k$ we can write $c_i = C_i + O(\varepsilon)$ where C_i are independent of ε.

6.3   Proof of Theorem 2

Finally, we need to estimate the difference $|\mathcal{M}_\varepsilon(\mathbf{T},F) - \mathcal{M}(\mathbf{T},F)|$. Now we apply the mean value theorem to conclude

$$|\mathcal{M}_\varepsilon(\mathbf{T},F) - \mathcal{M}(\mathbf{T},F)| \ll \varepsilon\|F\|_{1,\infty}.$$

Hence we choose ε to maximize the error terms, and note that $m_\mathbf{T}(\boldsymbol{\lambda}^{(0)}) \asymp 1$, leading to

$$\varepsilon^{-1}T_1^{-1}T_2^{-1}\log T_1 \log T_2 \|F\|_\Gamma = \varepsilon\|F\|_{1,\infty}.$$

Solving for ε then yields

$$\varepsilon = T_1^{-1/2}T_2^{-1/2}(\log T_1 \log T_2)^{1/2}\|F\|_\Gamma^{1/2}\|F\|_{1,\infty}^{-1/2}.$$

7   Expanding Horospheres $\mathrm{SL}_n(\mathbb{R})$

Finally, we extend our proof to all $n \geq 2$. For for general n the proof is almost identical to the proof for $\mathrm{SL}_3(\mathbb{R})$; the only essential input is a particular structure of the Casimir operators, $\Delta_1,\ldots,\Delta_{n-1}$ summarized in the following theorem:

Theorem 10 (Structure Theorem of Casimir Operators). *For $\mathrm{SL}_n(\mathbb{R})$, when acting on a right K-invariant function, $f \in L^2(\Gamma\backslash\mathcal{H})$, the Laplace-Beltrami operator satisfies*

$$\int_{\Gamma\backslash H} \Delta_1 f(ha_\mathbf{y})\mathrm{d}h = \left(\sum_{i=1}^{n-1} y_i^2 \partial_{y_1 y_1} - \sum_{i=1}^{n-2} y_i y_{i+1}\partial_{y_i y_{i+1}}\right)\int_{\Gamma\backslash H} f(ha_\mathbf{y})\mathrm{d}h. \quad (7.1)$$

Furthermore, for each Casimir operator, Δ_i there exists a differential operator D_i in the $\mathbf{y}$ variables, such that

$$\int_{\Gamma\backslash H} \Delta_i f(ha_\mathbf{y})\mathrm{d}h = D_i \int_{\Gamma\backslash H} f(ha_\mathbf{y})\mathrm{d}h. \tag{7.2}$$

Proof. We begin with an example for $n = 4$. In that case, if we integrate out the $\mathbf{x}$ coordinates, then using Goldfeld's `gln` Mathematica package [Gol06] we have that the Laplace-Beltrami operator, after integrating out the $\mathbf{x}$-directions, in $n = 4$ is given by

$$\Delta_1 f = \left(y_1^2 \partial_{y_1 y_1} + y_2^2 \partial_{y_2 y_2} + y_3^2 \partial_{y_3 y_3} - y_2 y_1 \partial_{y_1 y_2} - y_2 y_3 \partial_{y_2 y_3}\right) f.$$

The degree 3 operator is:

$$\begin{aligned}
\Delta_2 f = (&y_1^2 \partial_{y_1 y_1} - 3y_2 y_1^2 \partial_{y_1 y_1 y_2} - 7y_2 y_1 \partial_{y_1 y_2} + 3y_2^2 y_1 \partial_{y_1 y_2 y_2} - 2y_3^2 \partial_{y_3 y_3} \\
&- y_2 y_3 \partial_{y_2 y_3} + 3y_2 y_3^2 \partial_{y_2 y_3 y_3} + 4y_2^2 \partial_{y_2 y_2} - 3y_2^2 y_3 \partial_{y_2 y_2 y_3})f
\end{aligned}$$

and the degree 4 Casimir is then

$$\begin{aligned}
\Delta_3 f = (&y_1^4 \partial_{y_1}^4 + 4y_1^3 \partial_{y_1}^3 - 2y_1^3 y_2 \partial_{y_1}^3 \partial_{y_2} + 21y_1^2 \partial_{y_1}^2 - 12y_2 y_1^2 \partial_{y_1}^2 \partial_{y_2} + 3y_2^2 y_1^2 \partial_{y_2}^2 \partial_{y_1}^2 \\
&- 9y_2 y_1 \partial_{y_2} \partial_{y_1} + 6y_2^2 y_1 \partial_{y_1} \partial_{y_2}^2 - 2y_2^3 y_1 \partial_{y_2}^3 \partial_{y_1} - 3y_3^2 \partial_{y_3}^2 + 4y_3^3 \partial_{y_3}^3 + y_3^4 \partial_{y_3}^4 + 3y_2 y_3 \partial_{y_2 y_3} \\
&- 2y_2 y_3^3 \partial_{y_2} \partial_{y_3}^3 + 3y_2^2 \partial_{y_2}^2 - 6y_2^2 y_3 \partial_{y_2}^2 \partial_{y_3} + 3y_2^2 y_3^2 \partial_{y_2}^2 \partial_{y_3}^2 + 4y_2^3 \partial_{y_2}^3 - 2y_2^3 y_3 \partial_{y_2}^3 \partial_{y_3} + y_2^4 \partial_{y_2}^4)f.
\end{aligned}$$

In general, we can calculate the Casimir operators as in [Gol06, Proposition 2.3.3]. Then analyzing the differential operators from [Gol06, Definition 2.2.1] and applying an inductive argument on their products is enough to prove the (7.1) and (7.2). That is, any element of the lie algebra, X we can associate a first order differential operator coming from

$$T_X f = \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{tX})\Big|_{t=0}.$$

Thus T_X has the form

$$T_X = \eta_1 \partial_{x_1} + \cdots + \eta_{n(n-1)/2} \partial_{x_{n(n-1)/2}} + \mu_1 \partial_{y_1} + \cdots + \mu_{n-1} \partial_{y_{n-1}}.$$

Since we are multiplying on the right, it is evident that for the differential operators associated to X_{y_k} the coefficients $\eta_i = 0$ for $i = 1, \ldots n(n-1)/2$ and the μ_j do not depend on $\mathbf{x}$.

Moreover for the differential operators associated to X_{x_i}, it is not hard to see (by matrix multiplication rules) that η_i does not depend on x_i and $\mu_i = 0$ for $i = 1, \ldots, n-1$. From here (7.2) follows via integration by parts. The same holds for the dual elements $\overline{X_{x_i}}$. The Laplace-Beltrami operator can be found in numerous places, and (7.2) follows from a similar matrix multiplication argument.

□

Once again, fix an $\varepsilon > 0$, the proof begins by thickening the $\mathbf{y}$ directions. Recall that the volume measure is

$$\mathrm{d}\mathbf{x} \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} \mathrm{d}y_k$$

Therefore let

$$\psi_{T_i,\varepsilon}^{(i)} = \varepsilon^{-1} \mathbb{1}([1/T - \varepsilon/T^{k(n-k)+1}, 1/T + \varepsilon/T^{k(n-k)+1}]).$$

For $z \in \mathcal{H}$ let

$$\psi_{\mathbf{T},\varepsilon}(z) := \prod_{i=1}^{n-1} \psi_{T_i,\varepsilon}^{(i)}(y_i)$$

and let $\Psi_{\mathbf{T},\varepsilon} := \sum_{\Gamma_H \backslash \Gamma} \psi_{\mathbf{T},\varepsilon}(\gamma z)$. Then our thickened average is

$$\mathcal{M}_\varepsilon(\mathbf{T}, F) := \int_{\mathcal{F}} F(z) \Psi_{\mathbf{T},\varepsilon}(z) \mathrm{d}z,$$

where $\mathcal{F}$ is a fundamental domain for Γ.

With that, we apply the same unfolding steps and set

$$f(\mathbf{y}) := \int_{\Gamma_H \backslash H} F(ha(\mathbf{y})) \mathrm{d}h$$

then the analogue of (6.1) and (6.2) is: for any $\boldsymbol{\lambda}$

$$D_i f(\mathbf{y}) = \int_{\Gamma_H \backslash H} (\Delta_i - \lambda_i) F(ha(\mathbf{y})) \mathrm{d}h, \tag{7.3}$$

for $i = 1, \ldots, n-1$. Which, by moving to the unitary dual and using the fact that the Casimir elements are diagonal, yields the analogue of (6.8), that there exist functions $K_{i,\mathbf{T}}(\boldsymbol{\lambda})$ and times $\mathbf{b}_i$ such that we can express

$$\mathcal{M}_\varepsilon(\mathbf{T}) = \sum_{i=1}^{L} K_{i,\mathbf{T}}(\boldsymbol{\lambda}) \mathcal{M}_\varepsilon(\mathbf{b}_i, F) + \sum_{i=1}^{n-1} O(\|(\Delta_i - \lambda_i)F\|)$$

recall that $L = n!$ denotes the number of solutions $\boldsymbol{\nu}$ associated to the point $\boldsymbol{\lambda}$. From whence we can derive the main identity

$$\Psi_{\varepsilon,\mathbf{T}} = \sum_{i=1}^{L} K_{i,\mathbf{T}}(\boldsymbol{\Delta}) \Psi_{\varepsilon,\mathbf{b}_i}. \tag{7.4}$$

Moreover we have that

$$K_{i,\mathbf{T}}(\boldsymbol{\lambda}) \ll \begin{cases} I_{\boldsymbol{\nu}_i}(\mathbf{T}) & \text{if } \boldsymbol{\nu}_i \in [0, 1/n)^{n-1} \\ I_{cont}(\mathbf{T}) \log(T_1) \cdots \log T_{n-1} & \text{if } \nu_{i,j} = 1/n + it_j \text{ for all } j. \end{cases}$$

From here, the remainder is a somewhat trivial generalization of Section 6. Namely, after applying the spectral decomposition and (7.4) we arrive at the thickened equidistribution result

$$\mathcal{M}_\varepsilon(T, F) = \sum_{i=0}^{k} c_i m_{\mathbf{T}}(\boldsymbol{\lambda}^{(i)}) + O(\varepsilon^{-(n-1)/2} (I_{cont}(\mathbf{T}))^{-1} \log T_1 \cdots \log T_{n-1} \|F\|_\Gamma). \qquad (7.5)$$

The mean value theorem argument yields an error of size

$$\varepsilon m_{\mathbf{T}}(\boldsymbol{\lambda}^{(0)}) \|F\|_{1,\infty}.$$

Setting these errors equal to each other, and using that $m_{\mathbf{T}}(\boldsymbol{\lambda}_0) \asymp 1$ we arrive at

$$\varepsilon = \left(I_{cont}(\mathbf{T}) \|F\|_\Gamma \|F\|_{1,\infty}^{-1} \log T_1 \cdots \log T_{n-1} \right)^{2/(n+1)}$$

Thus our error becomes

$$O\left(\left(I_{cont}(\mathbf{T}) \log T_1 \cdots \log T_{n-1} \right)^{2/(n+1)} \right) \|F\|_\Gamma^{2/(n+1)} \|F\|_{1,\infty}^{(n-1)/(n+1)}. \qquad (7.6)$$

Acknowledgements

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