AVERAGE VARIANCE BOUNDS FOR INTEGER POINTS ON THE SPHERE

CHRISTOPHER LUTSKO

Abstract. Consider the integer points lying on the sphere of fixed radius projected onto the unit sphere. Duke showed that, on congruence conditions for the radius squared, these points equidistribute. To further this study of equidistribution, we consider the variance of the number of points in a spherical cap. An asymptotic for this variance was conjectured by Bourgain-Rudnick-Sarnak. We prove an upper bound of the correct size on the average (over radii) of these variances.

1. Introduction

Assume $n \in \mathcal{N} := \{n \in \mathbb{N} \ : \ n \not\equiv 0, 4, 7 (\bmod 8)\}$ and let

$$\mathcal{E}(n) = \{\mathbf{x} \in \mathbb{Z}^3 \ : \ |\mathbf{x}|^2 = n\}.$$

Further, let

$$\widehat{\mathcal{E}}(n) := \frac{1}{\sqrt{n}}\mathcal{E}(n) \subset \mathbb{S}^2.$$

Assuming the generalized Riemann hypothesis, Linnik [Lin68] showed that these points equidistribute on the sphere. This was then proved unconditionally by Duke [Duk88] and Golubeva-Fomenko [GF90] following breakthrough work of Iwaniec [Iwa87].

Given this equidistribution result, a natural question one can ask is whether the fine-scale statistics of the points $\widehat{\mathcal{E}}(n)$ converge to what one expects for uniformly distributed random variables on the sphere. This is exactly what Bourgain, Rudnick and Sarnak [BRS17] ask in their seminal work on the subject. One of the statistics they consider is the variance which is defined as follows. Given a set $\Omega \subset \mathbb{S}^2$, let

$$Z(n, \Omega) := \#(\widehat{\mathcal{E}}(n) \cap \Omega),$$

in which case, if we let $N_n := \#\mathcal{E}_n$, then the variance is defined to be

$$\mathrm{Var}(\Omega, n) := \int_{\mathbb{S}^2} |Z(n, \Omega + \zeta) - N_n \sigma(\Omega)|^2 \, \mathrm{d}\sigma(\zeta),$$

where σ is the normalized area measure on the sphere. Bourgain-Rudnick-Sarnak posed the following conjecture:

Conjecture 1 ([BRS17, Conjecture 1.6])**.** *Let Ω_n be a sequence of spherical caps. If $N_n^{-1+\varepsilon} \ll \sigma(\Omega_n) \ll N_n^{-\varepsilon}$ as $n \to \infty$, with $n \in \mathcal{N}$, then*

$$\mathrm{Var}(\Omega_n, n) \sim N_n \sigma(\Omega_n). \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 11E16, 11F30.
Key words and phrases. Lattice points, L-series, Fourier coefficients of automorphic forms.

A spherical cap is a set $\Omega_n = \{\mathbf{x} \in \mathbb{S}^2 \ : \ d(\mathbf{x}, \mathbf{y}) \leq r_n\}$, where d denotes the distance on the sphere. The conjecture is also stated for annuli. While proving this conjecture in full appears to be out of reach, Bourgain, Rudnick and Sarnak establish an upper bound [BRS17, Theorem 1.7] (assuming the Lindelöf hypothesis for $\mathrm{GL}(2)/\mathbb{Q}$ L-functions) of

$$\mathrm{Var}(\Omega_n, n) \ll n^{\varepsilon} N_n \sigma(\Omega_n), \qquad \forall \varepsilon > 0, \tag{1.2}$$

provided n is square-free.

Following this work, Humphries and Radziwiłł [HR22] proved the conjecture by Bourgain-Rudnick-Sarnak, provided n is square-free, in the microscopic regime (where $\sigma(\Omega_n)$ is small), and where Ω_n is an annulus with large inner radius. The regime where $\sigma(\Omega_n) \sim n^{-\varepsilon}$ remains open and, as noted by Humphries-Radziwiłł, is equivalent to the Lindelöf hypothesis for a family of L-functions. Following this, Shubin proved a similar theorem assuming GRH for $\mathrm{GL}(2)/\mathbb{Q}$ [Shu23]. Moreover, Lubotzky, Phillips, and Sarnak [LPS86] and Ellenberg, Michel, and Venkatesh [EMV13], among many others, have studied similar equidistribution problems.

The purpose of this paper is to provide unconditional progress towards this conjecture without assuming n is square-free by averaging over the different levels n. For that, fix a spherical cap $\Omega_X \subset \mathbb{S}^2$ and let

$$\mathcal{A}_{X,H} := \frac{1}{H} \sum_{\substack{n \in \mathcal{N} \\ X \leq n \leq X+H}} \mathrm{Var}(\Omega_X, n).$$

Further, let $\mathcal{A}_X := \mathcal{A}_{1,X-1}$ denote the average from $n = 1$ to X. The following result is the main result in the paper

Theorem 2. *Fix an integer X and a spherical cap Ω_X with area $\sigma(\Omega_X) = cN_X^{\delta}$, where $-1 < \delta < 0$ and $c > 0$ a constant. Then we have*

$$\mathcal{A}_X \ll X^{1/2} \sigma(\Omega_X). \tag{1.3}$$

Moreover, for any $\frac{X^{3/4}}{\sigma(X)^{3/4}} < H < \infty$ we have

$$\mathcal{A}_{X,H} \ll X^{1/2} \sigma(\Omega_X). \tag{1.4}$$

Note that $X^{1/2-\varepsilon} \ll N_X \ll X^{1/2+\varepsilon}$, thus Theorem 2 supports Conjecture 1.

1.1. **Strategy of proof:** First, we smooth the variance. This provides us with better control in the weight aspect. Then, by decomposing the variance we can relate it to the Fourier coefficients of the theta sums associated to the spherical harmonics. With that, the average of variances is related to the average value of Fourier coefficients of some half-integer weight, holomorphic cusp forms. It's well known that such averages can be well estimated, however for our purpose we need to bound these averages uniformly in both the weight and in n. We achieve this by carefully analyzing a Rankin-Selberg type L-series.

1.2. **Smoothing.** Rather than work with the discrete count $Z(n, \Omega)$, it is more convenient to smooth the count, thus introducing a parameter, $\rho > 0$, which we can choose at the end to optimize our bounds. To that end, given $z, \zeta \in \mathbb{S}^2$ and $\Omega \subset \mathbb{S}^2$, let

$$\chi_\Omega(z, \zeta) : \begin{cases} 1 & \text{if } z \in \Omega + \zeta, \\ 0 & \text{otherwise,} \end{cases}$$

denote the indicator function of $\Omega + \zeta$. Further, let

$$k_\rho(z,\zeta) := \begin{cases} \frac{1}{2\pi(1-\cos\rho)} & \text{if } d(z,\zeta) < \rho \\ 0 & \text{otherwise.} \end{cases}$$

Now convolve k_ρ and χ_Ω:

$$k_\rho(\Omega,\zeta,z) = (\chi_\Omega(\cdot,\zeta) * k_\rho)(z) = \int_{\mathbb{S}^2} k_\rho(z,\xi)\chi_\Omega(\xi,\zeta)d\sigma(\xi).$$

The smooth count at a point is then

$$Z_\rho(n,\Omega+\zeta) = \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} k_\rho(\Omega,\zeta,\mathbf{x}),$$

and the smooth variance can be written as

$$\mathrm{Var}_\rho(\Omega,n) := \int_{\mathbb{S}^2} |Z_\rho(n,\Omega+\zeta) - N_n\sigma(\Omega)|^2 \, \mathrm{d}\sigma(\zeta).$$

Given a spherical cap Ω_X, let

$$A_{X,\rho} := \frac{1}{X} \sum_{\substack{n\in\mathcal{N} \\ n\in[1,X]}} \mathrm{Var}_\rho(\Omega_X,n).$$

Further, let

$$A_{X,H,\rho} := \frac{1}{H} \sum_{\substack{n\in\mathcal{N} \\ n\in[X,X+H]}} \mathrm{Var}_\rho(\Omega_X,n).$$

The following is a smooth version of Theorem 2,

Theorem 3. *Fix an integer X, a smoothing parameter $\rho > 0$, and a spherical cap Ω_X with area $\sigma(\Omega_X) = cN_X^\delta$, where $-1 < \delta < 0$ and $c > 0$ is a constant. Then, for any $\varepsilon > 0$ we have the bound*

$$A_{X,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + X^\varepsilon\sigma(\Omega_X)^{1/2}\rho^{-1/2+\varepsilon}. \tag{1.5}$$

Moreover

$$A_{X,H,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + \frac{X^{1+\varepsilon}}{H}\sigma(\Omega_X)^{1/2}\rho^{-1/2-\varepsilon}. \tag{1.6}$$

With Theorem 3 at hand, Theorem 2 follows somewhat immediately:

Proof of Theorem 2. Assuming Theorem 3, to prove Theorem 2 we need to relate the variance to the smoothed variance. Consider the variance:

$$\begin{aligned}\mathrm{Var}(\Omega,n) &= \int_{\mathbb{S}^2} |Z(n,\Omega+\zeta) - \sigma(\Omega)N_n|^2 \, \mathrm{d}\sigma(\zeta) \\ &\leq \mathrm{Var}_\rho(\Omega,n) + \int_{\mathbb{S}^2} |Z(n,\Omega+\zeta) - Z_\rho(n,\Omega+\zeta)|^2 \, \mathrm{d}\sigma(\zeta).\end{aligned}$$

The latter quantity can then be written

$$\begin{aligned}
&\int_{\mathbb{S}^2} |Z(n,\Omega+\zeta) - Z_\rho(n,\Omega+\zeta)|^2 \,\mathrm{d}\sigma(\zeta)\\
&\qquad = \int_{\mathbb{S}^2} \left| \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} (\chi_\Omega(\mathbf{x},\zeta) - k_\rho(\Omega,\zeta,z)) \right| |Z_\rho(n,\Omega+\zeta)| \,\mathrm{d}\sigma(\zeta)\\
&\qquad \ll N_n\sigma(\Omega) \int_{\mathbb{S}^2} \left| \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \int_{\mathbb{S}^2} k_\rho(\mathbf{x},\xi) \left|\chi_\Omega(\mathbf{x},\zeta) - \chi_\Omega(\xi,\zeta)\right| \mathrm{d}\sigma(\xi) \right| \mathrm{d}\sigma(\zeta)\\
&\qquad = N_n\sigma(\Omega) \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} k_\rho(\mathbf{x},\xi) \left|\chi_\Omega(\mathbf{x},\zeta) - \chi_\Omega(\xi,\zeta)\right| \mathrm{d}\sigma(\xi)\mathrm{d}\sigma(\zeta)\\
&\qquad \ll \frac{N_n\sigma(\Omega)}{\rho^2} \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \int_{\mathbb{S}^2}\int_{d(\mathbf{x},\xi)<\rho} \left|\chi_\Omega(\mathbf{x},\zeta) - \chi_\Omega(\xi,\zeta)\right| \mathrm{d}\sigma(\xi)\mathrm{d}\sigma(\zeta)\\
&\qquad \ll N_n^2\sigma(\Omega)^{3/2}\rho.
\end{aligned}$$

Hence, we require $\rho \ll \frac{1}{X^{1/2+\varepsilon}\sigma(\Omega_X)^{1/2}}$.

Choosing $\rho = \frac{1}{X^{1/2+\varepsilon}\sigma(\Omega_X)^{1/2}}$ then yields

$$\begin{aligned}
A_X &\ll A_{X,\rho} + X^{1/2-\varepsilon}\sigma(\Omega_X)\\
&\ll X^{1/2}\sigma(\Omega_X) + \frac{1}{X^\varepsilon} + X^{1/4+\varepsilon}\sigma(\Omega_X)^{3/4}.
\end{aligned}$$

Which is the correct size if $X^{1/2}\sigma(\Omega_X) > X^{1/4+\varepsilon}\sigma(\Omega_X)^{3/4}$ or rather $\sigma(\Omega_X) > X^{-1+\varepsilon}$ which is always the case. When averaging over a window, equation (1.4) follows similar lines with the same choice of ρ.

□

2. Reducing to bounds on Fourier coefficients of automorphic forms

Consider the smooth variance:

$$\mathrm{Var}_\rho(\Omega_X, n) = \int_{\mathbb{S}^2} \left(Z_\rho(n;\Omega_X+\zeta) - N_n\sigma(\Omega_X)\right)^2 \mathrm{d}\sigma(\zeta).$$

We expand this into spherical harmonics: for $m = 0, 1, \dots$ and $k = m + 3/2$ let $\phi_{k,j}$ for $j = 1, 2, \dots, 2(k-1)$, denote an orthonormal basis of eigenfunctions for Δ on $\mathbb{S}^2$. Define the Weyl sum:

$$W_{k,j}(n) := \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \phi_{k,j}(\mathbf{x}). \tag{2.1}$$

Now, note that $k_\rho(\Omega_X,\zeta,z)$ is a function of $d(\zeta,z)$, let $k_\rho(t)$ denote the same function on $\mathbb{R}$. Then define:

$$h_X(k) := 2\pi \int_0^1 k_\rho(t) P_{k-3/2}(t)\mathrm{d}t,$$

where $P_{k-3/2}(t)$ is the $(k-3/2)^{th}$ Legendre polynomial. Then the smooth variance can be written as

$$\mathrm{Var}_\rho(\Omega_n, n) = \sum_k h_X(k)^2 \sum_{j=1}^{2(k-1)} |W_{k,j}(n)|^2 ,$$

where the sum over $k = m+3/2$ ranges over $m = 1, 2, \ldots$ (we use this notation throughout).

While [BRS17] use that the Weyl sums can be expressed in terms of special values of L-functions, we return to the original proof of Linnik's conjecture by Duke to express these Weyl sums in terms of Fourier coefficients of half integer weight modular forms.

Given a spherical harmonic, $\phi_{k,j}$, of degree $m = k-3/2$, let $\theta_{k,j}(z) := \sum_{\ell\in\mathbb{Z}^3} \phi_{k,j}(\ell)e(z|\ell|^2)$, then $\theta_{k,j}$ is a holomorphic cusp form of weight k for the group $\Gamma_0(4)$. Let $a_{k,j}(n)$ denote the n^{th} Fourier coefficient of $\theta_{k,j}$. Then

$$W_{k,j}(n) = \frac{a_{k,j}(n)}{n^{k/2-3/4}}$$

(see [Duk88] for more details).

With all that, we can express the average of variances as

$$A_{X,\rho} = \frac{1}{X} \sum_k h_X(m)^2 \sum_{j=1}^{2m+1} \sum_{\substack{n\in\mathcal{N}\\ n\in[1,X]}} \frac{|a_{k,j}(n)|^2}{n^{k-3/2}}.$$

To simplify matters, and since we only need an upper bound, we can complete the sum in n. That is, let

$$A_{X,\rho} \ll \frac{1}{X} \sum_k h_X(k)^2 \sum_{j=1}^{2m+1} \sum_{n\in[1,X]} \frac{|a_{k,j}(n)|^2}{n^{k-3/2}}.$$

2.1. **Fourier coefficients of holomorphic cusp forms.** Working more generally, let

$$f(z) = \sum_{i=1}^{\infty} a_{k,j}(n)e(nz)$$

be a holomorphic cusp form of weight k for $\Gamma_0(4)$. Then an immediate bound is the following

$$a_{k,j}(n) \ll_k n^{k/2-1/4}\tau(n), \tag{2.2}$$

where τ is the divisor function.

It is more convenient to normalize the Fourier coefficients by $n^{-(k-1)/2}$. Thus, let $b_{k,j}(n) := a_{k,j}(n)n^{-(k-1)/2}$. In which case, we want to bound the following

$$A_{X,\rho} \ll \frac{1}{X} \sum_k h_X(k)^2 \sum_{j=1}^{2(k-1)} \sum_{n\in[1,X]} n^{1/2} |b_{k,j}(n)|^2 .$$

By positivity, we then have that

$$A_{X,\rho} \ll \frac{1}{X^{1/2}} \sum_k h_X(k)^2 \mathcal{F}_X, \tag{2.3}$$

where

$$\mathcal{F}_X := \sum_{j=1}^{2(k-1)} \sum_{n\in[1,X]} |b_{k,j}(n)|^2 .$$

Hence the problem reduces to finding bounds on $\mathcal{F}_X$.

2.2. **Uniform bound.** Before bounding $\mathcal{F}_X$, we require the following lemma which gives a uniform bound on the Fourier coefficients $a_{k,j}(n)$, taking into account the n and k dependence, and the normalization. For a weight k modular form, f, we denote the L^2 mass by

$$\|f\|_2 = \left(\int_{\Gamma_0(4)\backslash\mathbb{H}} y^{k-2}\,|f(z)|^2\,\mathrm{d}x\mathrm{d}y\right)^{1/2} .$$

Lemma 4. *The Fourier coefficients of the theta series $\theta_{k,j}$ satisfy the following bound for any $\varepsilon > 0$ and any $j = 1, \ldots, 2(k-1)$. If $k < 2\pi en$ then*

$$\frac{|a_{k,j}(n)|^2}{\|\theta_{k,j}\|_2^2} \ll \frac{(4\pi)^k n^{k-1/2+\varepsilon}}{\Gamma(k-1)k^{1/2}}. \tag{2.4}$$

Otherwise, if $k > 2\pi en$ we have

$$\frac{|a_{k,j}(n)|^2}{\|\theta_{k,j}\|_2^2} \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}. \tag{2.5}$$

Proof. First, normalize $\theta_{k,j}$ by the weight k, L^2-mass. Now, $\theta_{k,j}/\|\theta_{k,j}\|_2$ can be taken to be an element of an orthonormal basis of $\mathcal{S}_k$, the space of weight k cusp forms. Let $\{g_\ell\}_{\ell=1}^L$ denote such a basis with $g_1 = \theta_{k,j}/\|\theta_{k,j}\|_2$. Then, using Petersson's formula for the Fourier coefficients of the Poincaré series in terms of generalized Kloosterman sums one can derive the following (see [Iwa87, Lemma 1])

$$\sum_{\ell=1}^{L} |\widehat{g_\ell}(n)|^2 = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\left(1 + 2\pi i^{-k} \sum_{c\equiv 0 (\mathrm{mod}\ 4)} c^{-1} J_{k-1}\left(\frac{4\pi n}{c}\right) K(n,n;c)\right), \tag{2.6}$$

where J_{k-1} is the Bessel function of order $k-1$ and K denotes the generalized Kloosterman sum:

$$K(a,b;c) := \sum_{d (\mathrm{mod}\ c)} \varepsilon_d^{-2k}\left(\frac{c}{d}\right) e\left(\frac{a\overline{d} + bd}{c}\right)$$

(see [Iwa87, 389] for the notation used in the above display). For our purposes the bound

$$|K(n,n;c)| \le (n,c)^{1/2} c^{1/2} \tau(c)$$

is all we need from K. As for the Bessel function, for $c < n/k$ we use that the Bessel function is bounded by 1, and for $c \ge n/k$ insert the bound $J_{k-1}(z) \le |z/2|^{k-1}/\Gamma(k)$ (see

[Bat53, p14 (4)]), to deduce

$$\begin{aligned}\sum_{\ell=1}^{L}|\widehat{g}_\ell(n)|^2 &\ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\left(1+\sum_{c=1}^{Y}c^{-1/2+\varepsilon}+\sum_{Y}^{\infty}c^{-1/2+\varepsilon}\left|2\pi n/c\right|^{k-1}/\Gamma(k)\right),\\ &\ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\left(1+Y^{1/2+\varepsilon}+\frac{(2\pi n)^{k-1}}{\Gamma(k)k}Y^{3/2-k+\varepsilon}\right),\end{aligned}$$

where $Y=\frac{n}{k}$. Applying Stirling's formula then yields (2.4).

When $k>2\pi en$ the same calculation yields

$$\sum_{\ell=1}^{L}|\widehat{g}_\ell(n)|^2 \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\frac{(2\pi n)^{k-1}}{\Gamma(k)}+\frac{(4\pi n)^{k-1}}{\Gamma(k-1)}.$$

Now applying Stirling's formula to $\Gamma(k)$ and using that $k>2\pi en$ we arrive at

$$\begin{aligned}\sum_{\ell=1}^{L}|\widehat{g}_\ell(n)|^2 &\ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\frac{(2\pi n)^{k-3/2}}{(k/e)^k}+\frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\\ &\ll \frac{(4\pi n)^{k-5/2}}{\Gamma(k-1)}+\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}.\end{aligned}$$

□

2.3. **L^2 bounds on $\theta_{k,j}$.** Furthermore, we will need the following bound on the L^2 norm of $\theta_{k,j}$:

Lemma 5. *For any $k\geq 5/2$ and any $j=1,\ldots 2(k-1)$, we have*

$$\|\theta_{k,j}\|_2^2=\int_{\Gamma_0(4)\backslash\mathbb{H}}y^{k-2}\left|\theta_{k,j}(z)\right|^2\mathrm{d}x\mathrm{d}y \ll \frac{\Gamma(k)}{(4\pi)^k}\left(|W_{k,j}(1)|^2+\frac{1}{2^{k-3/2-\varepsilon}}\right). \tag{2.7}$$

Moreover, summing over j yields

$$\sum_{j=1}^{2(k-1)}\|\theta_{k,j}\|_2^2 \ll \frac{\Gamma(k)}{(4\pi)^k}k. \tag{2.8}$$

Proof. The proof uses a standard trick to estimate L^2 norms of theta series. Namely, consider the Eisenstein series on $\Gamma_0(4)$:

$$E(z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\Im(\gamma z)^s,$$

where Γ_∞ is the stabalizer of ∞ in $\Gamma_0(4)$. Now consider the integral

$$\int_{\Gamma_0(4)\backslash\mathbb{H}}\left|\theta_{k,j}(z)\right|^2E(z,s)y^{k-2}\mathrm{d}x\mathrm{d}y.$$

Unfolding the Eisenstein series then yields

$$\int_{\Gamma_\infty\backslash\mathbb{H}}\left|\theta_{k,j}(z)\right|^2y^{s+k-2}\mathrm{d}x\mathrm{d}y.$$

We have that $\theta_{k,j}(z) = \sum_{\ell\in\mathbb{Z}^3} \phi_{k,j}(\ell)e(z|\ell|^2)$. Inserting this definition and using the orthogonality of exponentials with integral frequency we arrive at

$$\int_0^\infty \sum_{L\in\mathbb{N}} \sum_{\substack{\ell,\widetilde{\ell}\in\mathbb{Z}^3\\ |\ell|^2=|\widetilde{\ell}|^2=L}} \phi_{k,j}(\ell)\overline{\phi_{k,j}(\widetilde{\ell})}e^{-4\pi Ly}y^{s+k-2}\mathrm{d}y.$$

Applying the change of variables $y \mapsto 4\pi Ly$ we then have

$$\Gamma(s+k-1)\sum_{L\in\mathbb{N}} \frac{1}{(4\pi L)^{s+k-1}} \left| \sum_{\substack{\ell\in\mathbb{Z}^3\\ |\ell|^2=L}} \phi_{k,j}(\ell)\right|^2 .$$

Note that the sum being squared, is exactly the Weyl sum (2.1), thus we have

$$\Gamma(s+k-1)\sum_{L\in\mathcal{N}} \frac{1}{(4\pi L)^{s+k-1}} |W_{k,j}(L)|^2 .$$

Thus we arrive at the bound

$$\int_{\Gamma_0(4)\backslash\mathbb{H}} |\theta_{k,j}(z)|^2 E(z,s)y^{k-2}\mathrm{d}x\mathrm{d}y \ll \frac{\Gamma(s+k-1)}{(4\pi)^{s+k}} \sum_{L\in\mathcal{N}} \frac{1}{L^{s+k-1}} |W_{k,j}(L)|^2 .$$

Taking the residue at $s = 1$ yields the desired bound

$$\int_{\Gamma_0(4)\backslash\mathbb{H}} |\theta_{k,j}(z)|^2 y^{k-2}\mathrm{d}x\mathrm{d}y \ll \frac{\Gamma(k)}{(4\pi)^k} \sum_{L\in\mathcal{N}} \frac{1}{L^k} |W_{k,j}(L)|^2 .$$

Using the sup norm bound on spherical harmonics $\|\phi_{k,j}\|_\infty \ll k^{1/2}$ (see e.g [SW71, Chapter IV Cor. 2]) we can bound the Weyl sum by $W_{k,j}(L) \ll N_L k^{1/2} \ll L^{1/2+\varepsilon}k^{1/2}$ giving

$$\begin{aligned}\int_{\Gamma_0(4)\backslash\mathbb{H}} |\theta_{k,j}(z)|^2 y^{k-2}\mathrm{d}x\mathrm{d}y &\ll \frac{\Gamma(k)}{(4\pi)^k}\left(|W_{k,j}(1)|^2 + \sum_{L\geq 2}\frac{1}{L^{k-1/2-\varepsilon}}k\right)\\ &\ll \frac{\Gamma(k)}{(4\pi)^k}\left(|W_{k,j}(1)|^2 + \frac{1}{2^{k-3/2-\varepsilon}}\right).\end{aligned}$$

As for (2.8), summing (2.7) yields

$$\begin{aligned}\sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 &\ll \frac{\Gamma(k)}{(4\pi)^k}\left(\sum_{j=1}^{2(k-1)} |W_{k,j}(1)|^2 + \frac{k}{2^k}\right)\\ &\ll \frac{\Gamma(k)}{(4\pi)^k}\left(\sum_{j=1}^{2(k-1)} \left|\sum_{\mathbf{x}\in\widehat{\mathcal{E}}(1)} \phi_{k,j}(\mathbf{x})\right|^2 + \frac{k}{2^k}\right).\end{aligned}$$

To bound the level 1 Weyl sum, we have the following bound:

$$\sum_{j=1}^{2(k-1)} |\phi_{k,j}(z)|^2 = \frac{2(k-1)}{2\pi} \tag{2.9}$$

which, as stated in [LPS86, (2.11)] follows from the addition law for ultraspherical polynomials.

From there, (2.8) immediately from the fact that $\left|\widehat{\mathcal{E}}(1)\right| = 6$

$$\sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \ll \frac{\Gamma(k)}{(4\pi)^k}\left(\sum_{\mathbf{x}\in\widehat{\mathcal{E}}(1)} \sum_{j=1}^{2(k-1)} |\phi_{k,j}(\mathbf{x})|^2 + \frac{k}{2^k}\right) \ll \frac{\Gamma(k)}{(4\pi)^k} k$$

□

3. Bounds on L-series

Given our theta series $\theta_{k,j}(z)$, we wish to achieve bounds on the average of Fourier coefficients. To that end, define the normalized Dirichlet series

$$L_{k,j}(s) = \sum_{n\geq 1} \frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^s}.$$

Further, define the complete series as

$$\Lambda(s) = \zeta(2s)(2\pi)^{-2(s+k-1)}\gamma(k,s)L_{k,j}(s),$$

where the gamma factor is given by

$$\gamma(k,s) := \Gamma(s)\Gamma(s+k-1).$$

Then we have the following classical lemma

Lemma 6. *The complete series Λ can be meromorphically continued to $\mathbb{C}$ with a simple pole at $s = 1$. Moreover, it satisfies the functional equation*

$$\Lambda(s) = \Lambda(1-s). \tag{3.1}$$

Proof. Our aim is to use the standard Rankin-Selberg trick, namely, define the Eisenstein series

$$E(z,s) = \sum_{\Gamma_\infty\backslash\Gamma_0(4)} \Im(\gamma z)^s. \tag{3.2}$$

Then, by Parseval's identity, we have

$$(4\pi)^{-(s+k-1)} \sum_{n\geq 1} \frac{|a_{k,j}(n)|^2}{n^{s+k-1}}\Gamma(s+k-1) = \int_0^1\int_0^\infty |\theta_{k,j}(z)|^2 y^k y^s \frac{\mathrm{d}y}{y^2}.$$

Now note that by the modularity of the theta series, we have that $|f(z)|^2 y^k$ is $\Gamma_0(4)$-invariant. Thus, refolding the integral, we arrive at

$$(4\pi)^{-(s+k-1)} \sum_{n\geq 1} \frac{|a_{k,j}(n)|^2}{n^{s+k-1}}\Gamma(s+k-1) = \int_{\Gamma_0(4)\backslash\mathbb{H}} |\theta_{k,j}(z)|^2 \Im(z)^k E(z,s)\mu(\mathrm{d}z).$$

Now multiply both sides of the equation by $\pi^{-(s+k-1)}\Gamma(s)\zeta(2s)$. In which case, the functional equation for the Eisenstein series means that we obtain a functional equation for the L-series for $s \mapsto 1-s$.

The meromorphic continuation of the completed series can now be derived from the meromorphic continuation of the Eisenstein series (see for example [Iwa97] for a more in-depth discussion).

□

3.1. **Bounds on the critical line.** The next ingredient we will need is a bound on $L_{k,j}$ on the critical line $\Re(s) = 1/2$. For this, we will derive an approximate functional equation as done in [Blo20] which is itself motivated by [IK04, Theorem 5.3, Proposition 5.4], from which we derive the following bound

Lemma 7. *For any $\varepsilon > 0$, any $j = 1, \dots, 2(k-1)$, and for $|t| \ll k^{\varepsilon} X^{\varepsilon}$*

$$\left| L_{k,j}(\frac{1}{2} + it) \right| \ll \frac{(4\pi)^k}{\Gamma(k-1)} X^{\varepsilon} k^{1/2+\varepsilon}, \tag{3.3}$$

with the implicit constant depending only on ε.

Proof. Assume $0 < \Re(s) < 2$ and consider the integral

$$W_+(s) := \frac{1}{2\pi i} \int_{(1)} e^{u^2} \Lambda(u+s) \frac{\mathrm{d}u}{u}.$$

Shift the contour to (-1), thus picking up the residues at $u = 0$ and $u = 1 - s$:

$$W_+(s) = \frac{1}{2\pi i} \int_{(-1)} e^{u^2} \Lambda(u+s) \frac{\mathrm{d}u}{u} + \Lambda(s) + \frac{e^{(1-s)^2}}{1-s} \operatorname{Res}(\Lambda, 1).$$

Now, evaluate this expression at $s = \frac{1}{2} + it$, and apply the functional equation for Λ to the integral on $u = -1 + i\tau$, giving

$$\begin{aligned} \Lambda(s) &= W_+(s) + \frac{1}{2\pi i} \int_{(-1)} e^{u^2} \Lambda(-\frac{1}{2} + it + i\tau) \frac{\mathrm{d}u}{u} + \frac{e^{(1-s)^2}}{1-s} \operatorname{Res}(\Lambda, 1) \\ &= W_+(s) + W_-(s) + \frac{e^{(1-s)^2}}{1-s} \operatorname{Res}(\Lambda, 1), \end{aligned}$$

where

$$W_-(s) := \frac{1}{2\pi i} \int_{(-1)} e^{u^2} \Lambda(\frac{3}{2} + it + i\tau) \frac{\mathrm{d}u}{u}.$$

We calculate the residue in Lemma 5, giving us the bound

$$\Lambda(\frac{1}{2} + it) \ll W_+(s) + W_-(s) + \left| \frac{e^{(1/2-it)^2}}{1/2 - it} \right| \frac{1}{\pi^k}.$$

Now, consider

$$\frac{W_+(s)}{\gamma(k,s)} \ll \left| \sum_{n \geq 1} \frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^s} \int_{(1)} \frac{e^{u^2}}{n^u} \frac{\gamma(k, u+s)}{\gamma(k,s)} \zeta(2(u+s)) (2\pi)^{-2(u+s+k)} \frac{\mathrm{d}u}{u} \right|.$$

To bound this quantity we separate the sum over n into two regimes. Fix $T = \lceil 2\pi e k \rceil$, W_1 will sum over $n < T$ and W_2 will sum over $n \geq T$.

Consider first

$$W_2(s) := \left| \sum_{n > k^{1+\varepsilon}} \frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^s} \int_{(1)} \frac{e^{u^2}}{n^u} \frac{\gamma(k, u+s)}{\gamma(k,s)} \zeta(2(u+s)) (2\pi)^{-2(u+s+k)} \frac{\mathrm{d}u}{u} \right|.$$

Shift the contour to $\Re(u) = A$. Then we have

$$W_2(s) := \left| \sum_{n > T} \frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^s} \int_{\mathbb{R}} \frac{e^{A^2 - \tau^2 + 2iA\tau}}{n^{A+i\tau}} \frac{\gamma(k, A+i\tau+s)}{\gamma(k,s)} \zeta(2(A+i\tau+s)) (2\pi)^{-2(A+i\tau+s+k)} \frac{\mathrm{d}\tau}{A+i\tau} \right|.$$

The exponential causes the integral to be rapidly converging, thus we can bound $\tau \ll (k+|t|)^\varepsilon$. Moreover we can bound the zeta function by $(\tau+|t|)^\varepsilon \ll (|t|+k)^\varepsilon$, and bound the gamma factor using Stirling's formula:

$$\frac{\gamma(k, A+i\tau+s)}{\gamma(k,s)} \ll \exp(\pi|A+i\tau|)(|s|+3)^A(|s+k|+3)^A \tag{3.4}$$

(see [IK04, p 100]). Thus we arrive at the bound

$$\begin{aligned} W_2(s) &\ll_A (k+|t|)^\varepsilon(2\pi)^{-2k}\sum_{n>T}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^{1/2}}\int_{|\tau|\le(k+|t|)^\varepsilon}\left|\frac{e^{-\tau^2}}{n^A}\frac{\gamma(k,A+i\tau+s)}{\gamma(k,s)}\right|\mathrm{d}\tau \\ &\ll_A (k+|t|)^\varepsilon(1+|t|)^A(2\pi)^{-2k}\sum_{n>T}\frac{|b_{k,j}(n)|^2\,k^A}{\|\theta_{k,j}\|^2 n^{A+1/2}}. \end{aligned}$$

From here we apply Lemma 4 to conclude:

$$\begin{aligned} \frac{W_2(\frac12+it)}{\zeta(1+2it)(2\pi)^{-k}} &\ll (kX)^\varepsilon\sum_{n>T}\frac{k^A}{n^{A+1/2}}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2} \\ &\ll \frac{(4\pi)^k}{\Gamma(k-1)k^{1/2}}(kX)^\varepsilon k^A\sum_{n>T}\frac{1}{n^{A-\varepsilon}} \\ &\ll \frac{(4\pi)^k}{\Gamma(k-1)}(kX)^\varepsilon k^{1/2+\varepsilon}. \end{aligned}$$

Turning to W_1 we shift the contour to $\Re(u)=\frac12+\sigma$. Hence

$$W_1(s) := \left|\sum_{n<T}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^s}\int_{(\frac12+\sigma)}\frac{e^{u^2}}{n^u}\frac{\gamma(k,u+s)}{\gamma(k,s)}\zeta(2(u+s))(2\pi)^{-2(u+s+k)}\frac{\mathrm{d}u}{u}\right|.$$

we can again use the oscillatory term, e^{u^2}, to bound the integral such that $\tau := \Im(u) \ll (|t|+k)^\varepsilon$

$$\begin{aligned} W_1(s) &\ll \sum_{n<T}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^{1/2}}\left|\int_{|\tau|<(|t|+k)^\varepsilon}\frac{e^{u^2}}{n^u}\frac{\gamma(k,u+s)}{\gamma(k,s)}\zeta(2(u+s))(2\pi)^{-2(u+s+k)}\mathrm{d}\tau\right| \\ &\ll (2\pi)^{-2k}(|t|+k)^\varepsilon\sum_{n<T}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^{1+\sigma}}\int_{|\tau|<(|t|+k)^\varepsilon}e^{u^2}\frac{\gamma(k,u+s)}{\gamma(k,s)}\mathrm{d}\tau \\ &\ll (2\pi)^{-2k}(|t|+k)^\varepsilon(1+|t|)^{1/2+\sigma}\,|k|^{1/2+\sigma}\sum_{n<T}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2 n^{1+\sigma}}. \end{aligned}$$

From here we again apply Lemma 4 to conclude:

$$\begin{aligned} \frac{W_1(\frac12+it)}{\zeta(1+2it)(2\pi)^{-k}} &\ll (1+|t|)^{1/2+\sigma+\varepsilon}\,|k|^{1/2+\sigma+\varepsilon}\sum_{n<T}\frac{1}{n^{1+\sigma}}\frac{|b_{k,j}(n)|^2}{\|\theta_{k,j}\|^2} \\ &\ll (Xk)^\varepsilon\,|k|^{1/2+\sigma+\varepsilon}\,\frac{(4\pi)^{k-1}}{\Gamma(k-1)}. \end{aligned}$$

Set $\sigma=\varepsilon$ to yield the desired bound.

Finally, we note that W_- can be similarly bounded using the same technique. Together this yields (3.3).

□

3.2. **Bounds on $\mathcal{F}_X$.** To achieve the desired bound on $\mathcal{F}_X$ the following lemma allows us to express $\mathcal{F}_X$ in terms of the residue of $L_{k,j}$ at $s=1$.

Lemma 8. *For $\mathcal{F}_X$ as above, as $X \to \infty$ we have*

$$\mathcal{F}_X = \sum_{j=1}^{2(k-1)} C \frac{\|\theta_{k,j}\|_2^2}{(4\pi)^{-k}\Gamma(k)} X + O(X^{1/2+\varepsilon} k^{5/2+\varepsilon}). \tag{3.5}$$

Proof. Since we are only interested in an upper bound, we can smooth the sum in n. Thus let ψ be a smooth function, compactly supported on $[0,2]$, equal to 1 on $[0,1]$, then one has

$$\mathcal{F}_X \ll \sum_{j=1}^{2(k-1)} \sum_{n=1}^{\infty} |b_{k,j}(n)|^2 \, \psi\left(\frac{n}{X}\right).$$

Let $s = c+it$ and assume $c>1$, then we can use Mellin inversion to write

$$\mathcal{F}_X \ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widehat{\psi}(s) X^s L_{k,j}(s) \mathrm{d}s,$$

where $\widehat{\psi}$ denotes the Mellin transform of ψ.

Pushing the contour to the left, we pick up the contribution from the pole at $s=1$. Thus

$$\mathcal{F}_X \ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \left(\mathrm{Res}(L_{k,j},1) X + \frac{1}{2\pi i} \int_{(1/2)} L_{k,j}(s) X^s \widehat{\psi}(s) \mathrm{d}s \right). \tag{3.6}$$

For the residue of $L_{k,j}$, we can use the method from Lemma 6 to write the residue as

$$\mathrm{Res}(L_{k,j},1) = C \frac{1}{(4\pi)^{-k}\Gamma(k)}. \tag{3.7}$$

Thus, it remains to estimate the error term

$$R := \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \int_{(1/2)} L_{k,j}(s) \widehat{\psi}(s) X^s \mathrm{d}s.$$

Recall that the regularity properties of ψ imply decay in the Mellin transform $\widehat{\psi}$ as $\Im(s) \to \infty$. Since ψ can be chosen to be arbitrarily smooth, we can use this to restrict the range of integration to $[-X^\varepsilon, X^\varepsilon]$ with only a negligible error. Hence we can write

$$R \ll X^{1/2} \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \int_{-X^\varepsilon}^{X^\varepsilon} \left| L_{k,j}(\frac{1}{2}+it) \widehat{\psi}(\frac{1}{2}+it) \right| \mathrm{d}t + o(1).$$

To bound the remaining contribution we insert our bound on the half-line (3.3)

$$\begin{aligned} R &\ll X^{1/2+\varepsilon} \left(\frac{(4\pi)^k}{\Gamma(k-1)} k^{1/2+\varepsilon} \right) \left(\sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \right) \int_0^{X^\varepsilon} \left| \widehat{\psi}\left(\frac{1}{2}+it\right) \right| \mathrm{d}t \\ &\ll X^{1/2+\varepsilon} \left(\frac{(4\pi)^k}{\Gamma(k-1)} k^{1/2+\varepsilon} \right) \left(\sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \right) \end{aligned}$$

Now, from Lemma 5, we have that $\sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \ll \frac{\Gamma(k)}{(4\pi)^k} k$. Thus, we conclude

$$R \ll X^{1/2} T k^{5/2+\varepsilon}.$$

□

3.3. **Proof of Theorem 3.** The advantage of having smoothed is that h_X satisfies the bound:

$$h_X(k) \ll k^{-3/2} \sigma(\Omega_X)^{1/4} \min\left(1, \frac{(\sin\rho)^{1/2}}{k^{3/2}(1-\cos\rho)}\right), \tag{3.8}$$

which is derived in [LPS86, (2.13)].

Recall that the average of variances can be written

$$A_{X,\rho} \ll \frac{1}{X^{1/2}} \sum_k h_X(k)^2 \mathcal{F}_X.$$

First, apply Lemma 8 to extract the main term:

$$A_{X,\rho} \ll \frac{1}{X^{1/2}} \sum_k h_X(k)^2 \left(\sum_{j=1}^{2(k-1)} \frac{\|\theta_{k,j}\|_2^2}{(4\pi)^{-k}\Gamma(k)} X + X^{1/2+\varepsilon} k^{5/2+\varepsilon} \right).$$

To bound the first term in the brackets apply Lemma 5, giving

$$\begin{aligned} \mathcal{M} &:= X^{1/2} \sum_k h_X(k)^2 \sum_{j=1}^{2(k-1)} \frac{\|\theta(\cdot, \phi_{k,j})\|_2^2}{(4\pi)^{-k}\Gamma(k)} \\ &\ll X^{1/2} \sum_k h_X(k)^2 \sum_{j=1}^{2(k-1)} (|W_{k,j}(1)|^2 + \frac{1}{2^k}). \end{aligned}$$

The term involving $W_{k,j}(1)$ is exactly $X^{1/2}\operatorname{Var}_\rho(\Omega_X, 1)$, which can be bounded by $X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2}$. The other term can be fairly easily bounded using (3.8) giving

$$\mathcal{M} = X^{1/2}\sigma(\Omega_X) + O\left(X^{1/2}\rho\sigma(\Omega_X)^{1/2}\right).$$

To bound the error term

$$\mathcal{E} := X^\varepsilon \sum_k h_X(k)^2 k^{5/2+\varepsilon},$$

again, insert the bounds (3.8)

$$\begin{aligned} \mathcal{E} &\ll X^\varepsilon \sigma(\Omega_X)^{1/2} \sum_k k^{-1/2+\varepsilon} \min\left(1, \frac{\sin(\rho)^{1/2}}{k^{3/2}(1-\cos\rho)}\right)^2 \\ &\ll X^\varepsilon \sigma(\Omega_X)^{1/2} \left(\sum_{k<1/\rho} k^{-1/2+\varepsilon} + \frac{1}{\rho^3} \sum_{k>1/\rho} \frac{1}{k^{7/2-\varepsilon}} \right) \\ &\ll X^\varepsilon \sigma(\Omega_X)^{1/2} \rho^{-1/2-\varepsilon}. \end{aligned}$$

Thus, in the end we arrive at the bound

$$A_{X,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + X^\varepsilon \sigma(\Omega_X)^{1/2} \rho^{-1/2+\varepsilon}.$$

□

3.4. **Averages over windows.** All that remains is to bound the smoothed average over the window $[X, X+H]$. The procedure is very similar, first write

$$A_{X,H,\rho} \ll \frac{X^{1/2}}{H} \sum_k h_X(k) \mathcal{F}_{X,H},$$

where

$$\mathcal{F}_{X,H} = \sum_{j=1}^{2(k-1)} \sum_{n \in [X, X+H]} |b_{k,j}(n)|^2 .$$

Now following the proof of Lemma 8 we introduce a bump function ψ supported on $[-1/2, 3/2]$ which is equal to 1 on $[0,1]$. Then we can bound $\mathcal{F}_{X,H}$ by

$$\mathcal{F}_{X,H} \ll \sum_{j=1}^{2(k-1)} \sum_{n=1}^{\infty} |b_{k,j}(n)|^2 \, \psi\left(\frac{n-X}{H}\right).$$

Let $\widehat{\psi_{\frac{X}{H}}}$ be the Mellin transform of the function which takes $z \mapsto \psi(z - \frac{X}{H})$. Then

$$\mathcal{F}_{X,H} \ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widehat{\psi_{\frac{X}{H}}}(s) H^s L_{k,j}(s) \mathrm{d}s \right| .$$

Now push the contour to the left, picking up the residue at $s=1$ yielding

$$\begin{aligned}
\mathcal{F}_{X,H} &\ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \left(\mathrm{Res}(L_{k,j}, 1) H + \left| \frac{1}{2\pi i} \int_{(1/2)} L_{k,j}(s) H^s \widehat{\psi_{\frac{X}{H}}}(s) \mathrm{d}s \right| \right) \\
&\ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \left(\frac{H}{(4\pi)^{-k}\Gamma(k)} + \left| \frac{1}{2\pi i} \int_{(1/2)} L_{k,j}(s) H^s \widehat{\psi_{\frac{X}{H}}}(s) \mathrm{d}s \right| \right).
\end{aligned}$$

Once again, we can choose ψ to ensure sufficient regularity to restrict the integral to the range $[-X^\varepsilon, X^\varepsilon]$, yielding

$$\mathcal{F}_{X,H} \ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \left(\frac{H}{(4\pi)^{-k}\Gamma(k)} + H^{1/2} \int_{-X^\varepsilon}^{X^\varepsilon} \left| L_{k,j}(\frac{1}{2}+it) \right| \left| \widehat{\psi_{\frac{X}{H}}}(\frac{1}{2}+it) \right| \mathrm{d}t \right).$$

Now, apply Lemma 3.3 to the L-series and the bound $\left| \widehat{\psi_{\frac{X}{H}}}(\frac{1}{2}+it) \right| \ll \left(\frac{X}{H}\right)^{1/2}$

$$\mathcal{F}_{X,H} \ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \left(\frac{H}{(4\pi)^{-k}\Gamma(k)} + X^{1/2+\varepsilon} \frac{(4\pi)^k}{\Gamma(k-1)} k^{1/2+\varepsilon} \right).$$

Applying Lemma 5 now yields

$$\mathcal{F}_{X,H} \ll \sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \frac{H}{(4\pi)^{-k}\Gamma(k)} + X^{1/2+\varepsilon} k^{5/2+\varepsilon}.$$

To conclude, we find that

$$A_{X,H,\rho} \ll \frac{X^{1/2}}{H} \sum_k h_X(k) \left(\sum_{j=1}^{2(k-1)} \|\theta_{k,j}\|_2^2 \frac{H}{(4\pi)^{-k}\Gamma(k)} + X^{1/2+\varepsilon} k^{5/2+\varepsilon} \right)$$

In which case, the main term corresponds to the first term in the brackets:

$$A_{X,H,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X) + \frac{X^{1+\varepsilon}}{H} \sum_k h_X(m) m^{5/2+\varepsilon}.$$

The remaining term can now be bounded following same steps as in Section 3.3, hence we conclude

$$A_{X,H,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X) + \frac{X^{1+\varepsilon}}{H} \sigma(\Omega_X)^{1/2} \rho^{-1/2-\varepsilon}.$$

□

Acknowledgement

I am grateful to Valentin Blomer, Claire Burrin, Peter Humphries, Philippe Michel, and Zeev Rudnick, all of whom gave insightful comments on earlier drafts.

References

[Bat53] H. Bateman. *Higher transcendental functions*, volume II. New York-Toronto-London: McGraw-Hill, 1953.

[Blo20] V. Blomer. Epstein zeta-functions, subconvexity, and the purity conjecture. *J. Inst. Math. Jussieu*, 19(2):581–596, 2020.

[BRS17] J. Bourgain, Z. Rudnick, and P. Sarnak. Spatial statistics for lattice points on the sphere I: Individual results. *Bull. Iranian Math. Soc.*, 43(4):361–386, 2017.

[Duk88] W. Duke. Hyperbolic distribution problems and half-integral weight Maass forms. *Invent. Math.*, 92(1):73–90, 1988.

[EMV13] J. Ellenberg, P. Michel, and A. Venkatesh. Linnik's ergodic method and the distribution of integer points on spheres. In *Automorphic representations and L-functions*, volume 22 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 119–185. Tata Inst. Fund. Res., Mumbai, 2013.

[GF90] E. Golubeva and M Fomenko. Asymptotic distribution of integral points on the three-dimensional sphere. *Journal of Soviet Mathematics*, 52(3):3036–3048, 1990.

[HR22] P. Humphries and M. Radziwiłł. Optimal small scale equidistribution of lattice points on the sphere, Heegner points, and closed geodesics. *Comm. Pure Appl. Math.*, 75(9):1936–1996, 2022.

[IK04] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Iwa87] H. Iwaniec. Fourier coefficients of modular forms of half-integral weight. *Invent. Math.*, 87(2):385–401, 1987.

[Iwa97] H. Iwaniec. *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[Lin68] Yu. V. Linnik. *Ergodic properties of algebraic fields*, volume Band 45 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag New York, Inc., New York, 1968. Translated from the Russian by M. S. Keane.

[LPS86] A. Lubotzky, R. Phillips, and P. Sarnak. Hecke operators and distributing points on the sphere. I. volume 39, pages S149–S186. 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[Shu23] A. Shubin. Variance estimates in Linnik's problem. *Int. Math. Res. Not. IMRN*, 18:15425–15474, 2023.

[SW71] E. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*, volume No. 32 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1971.

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
Email address: `christopher.lutsko@uzh.ch`